

Capturing k -ary Existential Second Order Logic with k -ary Inclusion-Exclusion Logic

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Abstract

In this paper we analyze k -ary inclusion-exclusion logic, $\text{INEX}[k]$, which is obtained by extending first order logic with k -ary inclusion and exclusion atoms. We show that every formula of $\text{INEX}[k]$ can be expressed with a formula of k -ary existential second order logic, $\text{ESO}[k]$. Conversely, every formula of $\text{ESO}[k]$ with at most k -ary free relation variables can be expressed with a formula of $\text{INEX}[k]$. From this it follows that, on the level of sentences, $\text{INEX}[k]$ captures the expressive power of $\text{ESO}[k]$.

We also introduce several useful operators that can be expressed in $\text{INEX}[k]$. In particular, we define inclusion and exclusion quantifiers and so-called term value preserving disjunction which is essential for the proofs of the main results in this paper. Furthermore, we present a novel method of relativization for team semantics and analyze the duality of inclusion and exclusion atoms.

Keywords:

Inclusion logic, exclusion logic, dependence logic, team semantics, existential second order logic, expressive power

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1. Introduction

The origin of inclusion and exclusion logics lies in the notion of dependence and imperfect information in logic. First approaches in this area were *partially ordered quantifiers* by Henkin [10] and *IF-logic* (independence friendly logic) by Hintikka and Sandu [11]. The truth for IF-logic was originally defined by using semantic games of imperfect information ([12]), but an equivalent compositional semantics was presented later by Hodges [13]. However, in the compositional approach it is not sufficient to consider single assignments, but instead sets of assignments which are called *teams*.

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Teams can be seen as parallel positions in a semantic game, or can be interpreted as information sets or as databases ([18]). By using similar *team semantics* as Hodges, Väänänen [18] introduced *dependence logic* which extends first order logic with new atomic formulas called *dependence atoms*. Later Grädel and Väänänen [7] presented *independence logic* by analogously adding *independence atoms* to first order logic. The truth conditions for these atoms are defined by dependencies/independencies of the values of terms in a team. These logics have been recently studied actively with an attempt to formalize the dependency phenomena in different fields of science. There has been research in several areas such as database dependency theory ([15]), belief presentation ([3]) and quantum mechanics ([14]).

Inclusion and exclusion logics were first presented by Galliani [4]. They extend first order logic with *inclusion and exclusion atoms* as dependence atoms in dependence logic. Suppose that \vec{t}_1, \vec{t}_2 are k -tuples of terms and X is a team. The k -ary inclusion atom $\vec{t}_1 \subseteq \vec{t}_2$ says that the values of \vec{t}_1 are included in the values of \vec{t}_2 in the team X . The k -ary exclusion atom $\vec{t}_1 \mid \vec{t}_2$ analogously says that \vec{t}_1 and \vec{t}_2 get distinct values in X . These are simple and natural dependencies in database theory ([4]), and thus it is reasonable to consider such atoms in a team semantical setting.

Inclusion and exclusion atoms have some natural complementary properties. Exclusion logic is known to be *closed downwards* ([4]), i.e. if a team satisfies some formula, then also all of its subteams satisfy it. Inclusion logic, on the other hand, is known to be *closed under unions* ([4]), i.e. if each team in a set of teams satisfies a formula, then also their union satisfies it. However, neither of these logics is both closed downwards and under unions. Therefore the combination of these logics, *inclusion-exclusion logic*, has neither of these properties.

Exclusion logic is equivalent with dependence logic ([4]) which captures *existential second order logic*, ESO, on the level of sentences ([18]). Inclusion logic is not comparable with dependence logic in general ([4]), but captures *positive greatest fixed point logic* on the level of sentences, as shown by Galliani and Hella [6]. Hence exclusion logic captures NP, and inclusion logic captures PTIME over finite structures with linear order. Inclusion-exclusion logic has been shown to be equivalent with independence logic by Galliani [4]. Galliani has also shown in [4] that with inclusion-exclusion logic it is possible to define exactly those properties of teams which are definable in ESO. Thus we can say that inclusion-exclusion logic captures ESO *on the level of formulas*.

By these earlier results, we see that the expressive power of inclusion-exclusion logic is rather strong. Instead of studying this whole logic, we will consider its weaker fragments. One of the most canonical approaches is to restrict the arities of inclusion and exclusion atoms. In particular, unary atoms

are much simpler than inclusion and exclusion atoms in general. Hannula [8] has shown that inclusion logic has a strict arity hierarchy over graphs, but it is still open what is the exact fragment of ESO that corresponds to k -ary inclusion logic, $\text{INC}[k]$. Before our work similar research has not been done for exclusion- nor for inclusion-exclusion logic. Our main research question for this paper was to examine whether there is some natural fragment of ESO that corresponds to *unary* inclusion-exclusion logic, $\text{INEX}[1]$.

Similar research has been done on the related logics: Durand and Kontinen [2] have shown that, on the level of sentences, k -ary dependence logic captures the fragment of ESO in which at most $(k-1)$ -ary functions can be quantified. Galliani, Hannula and Kontinen [5] have shown that the same result holds also for k -ary independence logic. The arity hierarchy of ESO (over arbitrary vocabulary) is known to be strict, as shown by Ajtai [1] in 1983. Consequently dependence and independence logics have a strict arity hierarchy over sentences.

These earlier results, however, do not tell much about the expressive power of k -ary exclusion logic, $\text{EXC}[k]$, and k -ary inclusion-exclusion logic, $\text{INEX}[k]$, since the known translations from them to dependence and independence logics do not respect the arities of atoms. Also, since these results are proven on the level of sentences, we do not know much how does the arity affect the expressive power of these logics on the level of formulas.

We will show in Subsection 4.1 that every formula of $\text{EXC}[k]$ can be expressed with a formula of k -ary ESO, $\text{ESO}[k]$. The idea of this compositional translation is that for each occurrence of an exclusion atom $\vec{t}_1 \mid \vec{t}_2$ we quantify a separate k -ary relation variable that gives limits to the values that the tuple \vec{t}_1 can get and \vec{t}_2 cannot. We can formulate a similar, yet more complex, translation for $\text{INC}[k]$ and then merge these two translations to create a translation from $\text{INEX}[k]$ to $\text{ESO}[k]$.

In Subsection 4.2 we will show that all $\text{ESO}[k]$ -formulas that contain at most k -ary free relation variables can be expressed with a formula of $\text{INEX}[k]$. The translation we use here is compositional, very natural and uses inclusion and exclusion atoms in a dualistic way: The quantified k -ary relation variables P_i are just replaced with k -tuples \vec{w}_i of quantified first order variables. Then we simply replace atomic formulas of the form $P_i \vec{t}$ with inclusion atoms $\vec{t} \subseteq \vec{w}_i$ and formulas of the form $\neg P_i \vec{t}$ with exclusion atoms $\vec{t} \mid \vec{w}_i$.

In order to get make this last translation compositional, we also need a new operator called *term value preserving disjunction* which is introduced in Subsection 3.4. We will show that this operator can be expressed with inclusion and exclusion atoms, and furthermore when preserving values of k -tuples, it can be defined in $\text{INEX}[k]$. We will also explain in Subsection 3.4 why this is a useful operator for the framework of team semantics in general.

From our results it follows that, on the level of sentences, $\text{INEX}[k]$ captures

the expressive power of $\text{ESO}[k]$. In particular, by using only unary inclusion and exclusion atoms we get the expressive power of *existential monadic second order logic*, EMSO. This special case should be noted for the following reason: As a consequence of the results mentioned above ([2, 5]), if we extend FO with 1-ary dependence (or independence) atoms, the expressive power stays inside FO. But if we extend FO with 2-ary dependence (or independence) atoms, the expressive power becomes already stronger than EMSO. Thus INEX[1] deserves extra recognition by capturing this important fragment of ESO that has not yet been characterized in the framework of team semantics.

In addition to our main results, we also analyze the nature of inclusion and exclusion logics and their relationship more deeply. Even though inclusion and exclusion atoms are not contradictory negations of each other, we claim that they can be seen as duals of each other and thus they make a natural pair. This is one more reason why inclusion-exclusion logic can be seen as a quite canonical logic for the framework of team semantics.

We also analyze inclusion and exclusion relations from an another perspective by introducing *inclusion and exclusion quantifiers*. This can be seen as a step back to the origin of these logics, since dependence logic was inspired by IF-logic, in which dependencies were handled with quantification. In Subsection 3.2 we first define natural semantics for inclusion and exclusion quantifiers and then show that we can express them in inclusion-exclusion logic. We also show reversely that, by extending first order logic with these quantifiers, we obtain an equivalent logic with inclusion-exclusion logic. However, there are still some small, yet intriguing, differences between these two approaches.

By using several of our new operators – term value preserving disjunction and both existential and universal inclusion quantifiers – we can introduce a novel method of *relativization* for team semantics. This technique is introduced in Subsection 3.5 and later, in Section 5, we present further examples on how it can be applied. In Section 5 we also present some other concrete examples where we show how to use our translations and new operators to express some classical properties of models and teams in a rather straightforward way.

The structure of this paper is as follows: In Section 2 we review team semantics for FO and define inclusion and exclusion logics. In Section 3 we define several useful operators for inclusion-exclusion logic – such as inclusion and exclusion quantifiers and term value preserving disjunction. In Section 4 we present our translations between $\text{INEX}[k]$ and $\text{ESO}[k]$, and in Section 5 we present some further examples. After the conclusion in Section 6, there is an appendix where we present a single long and technical proof that has been omitted from the main text. For some further technical details, see an extended version of this paper [17].

2. Preliminaries

In this section we first define team semantics for first order logic. Then we present inclusion and exclusion logics, define team semantics for them and review some of their known properties. For some further details, see [17].

2.1. Syntax and team semantics for first order logic

Let L be a vocabulary. We denote the set of L -terms by T_L . If $\vec{t} = t_1 \dots t_k$ and $t_i \in T_L$ for each $i \leq k$, we write $\vec{t} \in T_L$. The set of variables occurring in a term t is denoted by $\text{Vr}(t)$. For a tuple $\vec{t} = t_1 \dots t_k$ of L -terms we write $\text{Vr}(\vec{t}) := \text{Vr}(t_1) \cup \dots \cup \text{Vr}(t_k)$. The set of first order logic formulas with respect to vocabulary L , denoted by FO_L , is defined in the standard way – except that we require all formulas to be in *negation normal form*. FO_L -formulas of the form $t_1 = t_2$, $\neg t_1 = t_2$, $R\vec{t}$ and $\neg R\vec{t}$ are called *literals*. We denote the set of subformulas of an FO_L -formula φ by $\text{Sf}(\varphi)$, the set of variables occurring in φ by $\text{Vr}(\varphi)$ and the set of free variables of φ by $\text{Fr}(\varphi)$.

Let $\mathcal{M} = (M, \mathcal{I})$ be an L -model. An *assignment* s for M is a function that is defined in some set of variables, $\text{dom}(s)$, and ranges over M . A *team* X for M is any set of assignments for M with a common domain, denoted by $\text{dom}(X)$. In the literature usually only teams with finite domains have been considered, but for this paper there is no need to assume the domains of teams to be finite. Note that we also allow the empty assignment $s = \emptyset$ and the empty team $X = \emptyset$. For the empty team we allow *any of set variables* to be interpreted as its domain (this is practical for certain technical reasons). The empty team is not to be confused with the team $X = \{\emptyset\}$ which has a special role with FO_L -sentences.

Let s be an assignment, $\vec{a} := (a_1, \dots, a_k) \in M^k$ and $\vec{x} := x_1 \dots x_k$ a tuple of variables. The assignment $s[\vec{a}/\vec{x}]$ is defined in $\text{dom}(s) \cup \text{Vr}(\vec{x})$, and it maps a variable x_i to a_i ($i \leq k$), and all other variables as the assignment s . For a team X , a set $A \subseteq M^k$ and a function $\mathcal{F} : X \rightarrow \mathcal{P}(M^k)$ we write

$$\begin{aligned} X[A/\vec{x}] &:= \{s[\vec{a}/\vec{x}] \mid s \in X, \vec{a} \in A\} \\ X[\mathcal{F}/\vec{x}] &:= \{s[\vec{a}/\vec{x}] \mid s \in X, \vec{a} \in \mathcal{F}(s)\}. \end{aligned}$$

Let \mathcal{M} be an L -model, s an assignment and $t \in T_L$ s.t. $\text{Vr}(t) \subseteq \text{dom}(s)$. The interpretation of t with respect to \mathcal{M} and s , $t^{\mathcal{M}}\langle s \rangle$, is denoted simply by $s(t)$. Let $\vec{t} := t_1 \dots t_k \in T_L$ and let X be a team s.t. $\text{Vr}(\vec{t}) \subseteq \text{dom}(X)$. We write

$$s(\vec{t}) := (s(t_1), \dots, s(t_k)) \quad \text{and} \quad X(\vec{t}) := \{s(\vec{t}) \mid s \in X\}.$$

Note that $s(\vec{t})$ is a vector in M and $X(\vec{t})$ is a k -ary relation in M . We write $\mathcal{P}^*(A) := \mathcal{P}(A) \setminus \{\emptyset\}$. We are now ready to define *team semantics* for FO.

Definition 2.1. Let \mathcal{M} be an L -model, $\varphi \in \text{FO}_L$ and X a team such that $\text{Fr}(\varphi) \subseteq \text{dom}(X)$. We define the *truth of φ in \mathcal{M} and X* , denoted by $\mathcal{M} \models_X \varphi$:

- $\mathcal{M} \models_X t_1 = t_2$ iff $s(t_1) = s(t_2)$ for all $s \in X$.
- $\mathcal{M} \models_X \neg t_1 = t_2$ iff $s(t_1) \neq s(t_2)$ for all $s \in X$.
- $\mathcal{M} \models_X R\vec{t}$ iff $s(\vec{t}) \in R^{\mathcal{M}}$ for all $s \in X$.
- $\mathcal{M} \models_X \neg R\vec{t}$ iff $s(\vec{t}) \notin R^{\mathcal{M}}$ for all $s \in X$.
- $\mathcal{M} \models_X \psi \wedge \theta$ iff $\mathcal{M} \models_X \psi$ and $\mathcal{M} \models_X \theta$.
- $\mathcal{M} \models_X \psi \vee \theta$ iff there are $Y, Y' \subseteq X$ s.t. $Y \cup Y' = X$, $\mathcal{M} \models_Y \psi$ and $\mathcal{M} \models_{Y'} \theta$.
- $\mathcal{M} \models_X \exists x \psi$ iff there is $F : X \rightarrow \mathcal{P}^*(M)$ such that $\mathcal{M} \models_{X[F/x]} \psi$.
- $\mathcal{M} \models_X \forall x \psi$ iff $\mathcal{M} \models_{X[M/x]} \psi$.

Remark. In the truth definition above we introduced so-called *lax semantics* for existential quantifier. In this definition the quantified variable can be given several witnesses. From the perspective of game-theoretic semantics this can be interpreted as the verifying player having a *non-deterministic strategy* when choosing a value for the quantified variable ([3]). An alternative semantics, so-called *strict semantics*, is to allow only a single witness for each assignment. In first the order case these two truth definitions are equivalent¹ ([4]), but this does not hold when we extend FO with inclusion atoms.

For $\varphi \in \text{FO}_L$ and $\vec{x} := x_1 \dots x_k$, we write $\exists \vec{x} \varphi := \exists x_1 \dots \exists x_k \varphi$ and $\forall \vec{x} \varphi := \forall x_1 \dots \forall x_k \varphi$. By Definition 2.1, consecutive quantifications modify the team after the evaluation of each quantifier. Nevertheless, as shown by the following easy proposition, it is equivalent to quantify several elements in M one after another and to quantify a single vector in M .

Proposition 2.1. *For any k -tuple \vec{x} and $\varphi \in \text{FO}_L$ we have*

- a) $\mathcal{M} \models_X \exists \vec{x} \varphi$ iff there exists $\mathcal{F} : X \rightarrow \mathcal{P}^*(M^k)$ such that $\mathcal{M} \models_{X[\mathcal{F}/\vec{x}]} \varphi$.
- b) $\mathcal{M} \models_X \forall \vec{x} \varphi$ iff $\mathcal{M} \models_{X[M^k/\vec{x}]} \varphi$.

Note that with lax semantics for existential quantifier, when we quantify a k -tuple of variables, we can actually quantify a k -ary relation in M .

First order logic with team semantics has so-called *flatness*-property:

Proposition 2.2 ([18], Flatness). *Let X be a team and $\varphi \in \text{FO}_L$. Then*

$$\mathcal{M} \models_X \varphi \text{ iff } \mathcal{M} \models_{\{s\}} \varphi \text{ for all } s \in X.$$

¹Also note that, in the general case, the lax version is not stronger since we can always turn a strict quantifier into the corresponding lax quantifier by adding a “dummy” universal quantifier before it in the formula. That is, if z is a fresh variable, then the formula $\exists x \varphi$ has same truth condition with lax semantics as the formula $\forall z \exists x \varphi$ with strict semantics.

We write \models_s^T and \models^T for truth with the standard Tarski semantics. The following proposition shows how team semantics is related to Tarski semantics.

Proposition 2.3 ([18]). *Let $\varphi \in \text{FO}_L$ and let s be an assignment. Then for all FO_L -formulas we have $\mathcal{M} \models_s^T \varphi$ iff $\mathcal{M} \models_{\{s\}} \varphi$. In particular, for all FO_L -sentences, we have $\mathcal{M} \models^T \varphi$ iff $\mathcal{M} \models_{\{\emptyset\}} \varphi$.*

Note that, by flatness, $\mathcal{M} \models_X \varphi$ iff $\mathcal{M} \models_s^T \varphi$ for all $s \in X$. In this sense we can say that team semantics for FO is a generalization of Tarski semantics.

By Proposition 2.3 it is natural to write $\mathcal{M} \models \varphi$ when we mean $\mathcal{M} \models_{\{\emptyset\}} \varphi$. Note that $\mathcal{M} \models_{\emptyset} \varphi$ holds trivially for all FO_L -formulas φ by Definition 2.1. In general we say that any logic \mathcal{L} with team semantics has *empty team property* if $\mathcal{M} \models_{\emptyset} \varphi$ holds for all \mathcal{L} -formulas φ .

We say that a logic \mathcal{L} is *local* if the truth of formulas is determined only by the values of the free variables in a team, i.e. the following holds for all \mathcal{L} -formulas φ .

$$\mathcal{M} \models_X \varphi \text{ iff } \mathcal{M} \models_{X \upharpoonright \text{Fr}(\varphi)} \varphi,$$

where $X \upharpoonright \text{Fr}(\varphi) := \{s \upharpoonright \text{Fr}(\varphi) \mid s \in X\}$ and $s \upharpoonright \text{Fr}(\varphi)$ is an assignment such that $\text{dom}(s \upharpoonright \text{Fr}(\varphi)) = \text{Fr}(\varphi)$ and $(s \upharpoonright \text{Fr}(\varphi))(x) = s(x)$ for each $x \in \text{Fr}(\varphi)$. FO is clearly local by Propositions 2.2 and 2.3. We define two more important properties for any logic \mathcal{L} with team semantics.

Definition 2.2. Let \mathcal{L} be a logic with team semantics. We say that

- \mathcal{L} is *closed downwards* if the following implication holds:

$$\text{If } \mathcal{M} \models_X \varphi \text{ and } Y \subseteq X, \text{ then } \mathcal{M} \models_Y \varphi.$$

- \mathcal{L} is *closed under unions* if the following implication holds:

$$\text{If } \mathcal{M} \models_{X_i} \varphi \text{ for every } i \in I, \text{ then } \mathcal{M} \models_{\cup_{i \in I} X_i} \varphi.$$

By flatness, FO is both closed both downwards and under unions.

2.2. Inclusion and exclusion logics

Inclusion and exclusion logics are obtained by adding inclusion and exclusion atoms, respectively, to FO with team semantics. By allowing the use of the both of these atoms we get inclusion-exclusion logic which is our main topic of interest in this paper. We first present the syntax and semantics for inclusion logic (INC).

Definition 2.3. If \vec{t}_1, \vec{t}_2 are k -tuples of L -terms, $\vec{t}_1 \subseteq \vec{t}_2$ is a k -ary *inclusion atom*. The language INC_L is defined as FO_L , except that (non-negated) inclusion atoms – of any arity – may be used as literals.

Let \mathcal{M} be a model and X a team s.t. $\text{Vr}(\vec{t}_1\vec{t}_2) \subseteq \text{dom}(X)$. We define the truth of $\vec{t}_1 \subseteq \vec{t}_2$ in the model \mathcal{M} and the team X :

$$\mathcal{M} \models_X \vec{t}_1 \subseteq \vec{t}_2 \quad \text{iff for all } s \in X \text{ there exists } s' \in X \text{ s.t. } s(\vec{t}_1) = s'(\vec{t}_2).$$

This truth condition can be written equivalently as follows:

$$\mathcal{M} \models_X \vec{t}_1 \subseteq \vec{t}_2 \quad \text{iff } X(\vec{t}_1) \subseteq X(\vec{t}_2).$$

Example 2.1. Let $\vec{t}_1, \dots, \vec{t}_m$ be k -tuples of L -terms and \vec{x} a k -tuple of fresh variables. Now the following holds for all *nonempty* teams X :

$$\mathcal{M} \models_X \forall \vec{x} \left(\bigvee_{i \leq m} \vec{x} \subseteq \vec{t}_i \right) \quad \text{iff} \quad \bigcup_{i \leq m} X(\vec{t}_i) = M^k.$$

In particular, for $t \in T_L$ and $X \neq \emptyset$ we have $\mathcal{M} \models_X \forall x (x \subseteq t)$ iff $X(t) = M$. Note that this property is not closed downwards and thus it cannot be expressed in dependence logic (which is closed downwards as shown in [18]).

Next we present the syntax and semantics for exclusion logic (EXC).

Definition 2.4. If $\vec{t}_1, \vec{t}_2 \in T_L$ are k -tuples, $\vec{t}_1 \mid \vec{t}_2$ is a *k-ary exclusion atom*. The language EXC_L is defined using by exclusion atoms as literals in FO_L .

Let \mathcal{M} be a model and X a team s.t. $\text{Vr}(\vec{t}_1\vec{t}_2) \subseteq \text{dom}(X)$. We define the truth of $\vec{t}_1 \mid \vec{t}_2$ in the model \mathcal{M} and the team X :

$$\mathcal{M} \models_X \vec{t}_1 \mid \vec{t}_2 \quad \text{iff for all } s, s' \in X : s(\vec{t}_1) \neq s'(\vec{t}_2).$$

This truth condition can be written equivalently as follows:

$$\mathcal{M} \models_X \vec{t}_1 \mid \vec{t}_2 \quad \text{iff } X(\vec{t}_1) \cap X(\vec{t}_2) = \emptyset.$$

In inclusion-exclusion logic (INEX) we may use both inclusion and exclusion atoms, and the corresponding language is denoted by INEX_L . INC and EXC have both been shown local². By the truth definitions of inclusion and exclusion atoms, it is easy to see that INC and EXC both satisfy empty team property. Hence also INEX satisfies these properties. Neither inclusion nor exclusion logic has flatness-property. Galliani [4] has shown that INC is closed under unions,

²Exclusion logic has been shown equivalent with dependence logic ([4]) which is known to be local ([18]). Inclusion logic has been shown local by Galliani [4], but for this proof the lax semantics is required. With strict semantics the locality of INC is lost, which is one of the reasons why the lax semantics is considered to be a more natural choice to be used in team semantics. Inclusion logic with strict semantics has also been studied (see for example [9]).

but not downwards. On the other hand, EXC is closed downwards but not under unions ([18]). Hence INEX is not closed downwards nor under unions.

Definition 2.5. If $\varphi \in \text{INEX}_L$ contains at most k -ary inclusion and exclusion atoms, we say that φ is an $\text{INEX}_L[k]$ -formula. By allowing only the use of these formulas, we obtain k -ary inclusion-exclusion logic, denoted by $\text{INEX}[k]$. Furthermore, k -ary inclusion logic ($\text{INC}[k]$) and k -ary exclusion logic ($\text{EXC}[k]$) are defined analogously.

Note that the exclusion atom $\vec{t}_1 \mid \vec{t}_2$ is not the *contradictory negation* of the inclusion atom $\vec{t}_1 \subseteq \vec{t}_2$, and that the former is symmetric while the latter is not (that is, $\vec{t}_1 \mid \vec{t}_2 \equiv \vec{t}_2 \mid \vec{t}_1$ but $\vec{t}_1 \subseteq \vec{t}_2 \not\equiv \vec{t}_2 \subseteq \vec{t}_1$). The contradictory negations of k -ary inclusion and exclusion atoms can be defined in $\text{INEX}[k]$ for *nonempty* teams, as shown by the following example.

Example 2.2. Let \mathcal{M} be a model, X a nonempty team, $\vec{t}_1, \vec{t}_2 \in T_L$ k -tuples and \vec{x} a k -tuple of variables. It is easy to see that we have

$$\begin{aligned} \mathcal{M} \not\models_X \vec{t}_1 \mid \vec{t}_2 & \text{ iff } \mathcal{M} \models_X \exists \vec{x} (\vec{x} \subseteq \vec{t}_1 \wedge \vec{x} \subseteq \vec{t}_2) \\ \mathcal{M} \not\models_X \vec{t}_1 \subseteq \vec{t}_2 & \text{ iff } \mathcal{M} \models_X \exists \vec{x} (\vec{x} \subseteq \vec{t}_1 \wedge \vec{x} \mid \vec{t}_2). \end{aligned}$$

If we would use *negated* inclusion/exclusion atoms with the semantics of the contradictory negation in INEX, we would lose empty team property since the contradictory negations of these atoms are false in the empty team. But for nonempty teams, this extension would not give us any more expressive power.

Observation 2.1. In team semantics contradictory negation is not equivalent with the negation \neg that is used with literals. This is because, if φ is of the form $\vec{t}_1 = \vec{t}_2$ or $R\vec{t}$, the claims $\mathcal{M} \not\models_X \varphi$ and $\mathcal{M} \models_X \neg\varphi$ are not necessarily equivalent when $|X| > 1$. Since inclusion and exclusion atoms are atomic formulas as (non-negated) literals, their *negations* should behave similarly as the negations of literals. Therefore, if we would define negated inclusion or exclusion atoms, the semantics of contradictory negation would not be a natural choice for it. We will discuss further the issue of sensible semantics for negated atoms in the end of Section 4.

3. Defining new operators for inclusion-exclusion logic

In this section we will define several useful operators for $\text{INEX}[k]$. First we will define *constancy atoms* and *intuitionistic disjunction*. Then we will introduce *inclusion and exclusion quantifiers* which present a new approach to inclusion and exclusion dependencies. Then we define a new operator called *term value preserving disjunction* which will be essential for our translation from $\text{ESO}[k]$ to $\text{INEX}[k]$ in the next section. Finally we will introduce a method

called *relativization* that is an application which uses several of the new operators defined in this section.

3.1. Constancy atoms and intuitionistic disjunction

Constancy atom $=(t)$ is a unary dependence atom ([18]). It simply says that the term t has a constant value in a (nonempty) team. Galliani [4] has shown that this atom can be expressed by using unary exclusion atom. Thus we can define this atom as an abbreviation in $\text{INEX}_L[k]$ for any $k \geq 1$.

Definition 3.1 ([4]). Let $t \in T_L$ and x a fresh variable. We define *constancy atom* $=(t)$, as an abbreviation, as follows: $=(t) := \forall x (x = t \vee x \mid t)$.

Proposition 3.1 ([4]). *With the assumptions of the previous definition, we obtain the following truth condition: $\mathcal{M} \models_X =(t)$ iff $|X(t)| = 1$ or $X = \emptyset$.*

Intuitionistic disjunction \sqcup is obtained by lifting the Tarski semantics of disjunction from single assignments to teams. That is, $\varphi \sqcup \psi$ is true in a team X if either φ or ψ is true in X . Galliani [3] has shown that this operator can be expressed with constancy atoms in any logic with empty team property. We will define this operator in INEX here in the same way – with the addition of the special case of single element models.

Definition 3.2 ([3]). Let $\varphi, \psi \in \text{INEX}_L$. We define *intuitionistic disjunction* $\varphi \sqcup \psi$, as an abbreviation, in the following way:

$$\begin{aligned} \varphi \sqcup \psi := & \left(\gamma_{=1} \wedge (\varphi \vee \psi) \right) \vee \exists z_1 \exists z_2 \left(=(z_1) \wedge =(z_2) \right. \\ & \left. \wedge ((z_1 = z_2 \wedge \varphi) \vee (z_1 \neq z_2 \wedge \psi)) \right), \end{aligned}$$

where z_1, z_2 are fresh variables and $\gamma_{=1}$ is a shorthand for $\forall z_1 \forall z_2 (z_1 = z_2)$.

Proposition 3.2 ([3]). *With the assumptions of the previous definition, we obtain the following truth condition: $\mathcal{M} \models_X \varphi \sqcup \psi$ iff $\mathcal{M} \models_X \varphi$ or $\mathcal{M} \models_X \psi$.*

3.2. Inclusion and exclusion quantifiers

Here we will consider inclusion and exclusion relations from a new perspective. Instead of having atomic formulas that express them, we embed these relations to the truth conditions of quantifiers. By this approach, we are aiming to obtain a logic that has similar relationship with INEX , as there is between IF-logic and dependence logic. We will define inclusion and exclusion versions for both existential and universal quantifiers. We will also show that we can express them by using inclusion and exclusion atoms, and thus use them freely as abbreviations in INEX . Before giving the actual definitions, we first consider what kind of semantics would be intuitive for such operators.

In independence friendly logic we can use so-called IF-quantifiers which state that the values given for a quantified variable are independent of the values of certain other variables. It would be essentially equivalent to define “dependence friendly” quantifiers ([18]) which state that the values for a quantified variable is allowed to depend only on a certain set of variables. Dependence atoms of dependence logic ([18]) state a the same property about the values of variables in a team on an atomic level. We take a reverse approach here: Instead of stating that inclusion or exclusion relation holds for certain variables in a team, we say that inclusion or exclusion holds for a certain variable when it is quantified. Syntactically this would give us quantifiers of the form $(\exists x \subseteq y)$, $(\forall x \subseteq y)$, $(\exists x | y)$ and $(\forall x | y)$.

Remark. Since inclusion is not a symmetric relation, one could also consider semantics for quantifiers of the form $(\exists x \supseteq y)$ and $(\forall x \supseteq y)$. For the first one of these we see at least two non-equivalent natural semantical approaches, but the meaning of the latter one seems to become trivial. This question is not examined further in this paper, but a reader is encouraged to consider intuitive semantics for such quantifiers after reading this section.

Before considering natural semantics for these quantifiers, we introduce so-called *storing operator* that is needed in the definitions later. The idea for it is simply that we copy the values of a given tuple \vec{t} of terms into a given tuple \vec{u} of variables. This way it is possible to refer to the old values of \vec{t} , even if they change later in (re)quantifications.

Definition 3.3. Let $\varphi \in \text{INEX}_L$, $\vec{t} \in T_L$ a k -tuple and \vec{u} a k -tuple of variables. The \vec{t} to \vec{u} *storing operator*, $[\vec{t} \triangleright \vec{u}]$, is defined as:

$$[\vec{t} \triangleright \vec{u}]\varphi := \exists \vec{u} (\vec{u} = \vec{t} \wedge \varphi).$$

For this operator to work as desired, we need to set a requirement that the variables in the tuple \vec{u} do not occur in the tuple \vec{t} . However, naturally we must allow the variables in the tuple \vec{u} to be free variables in the formula φ .

The following lemma for storing operator is obvious.

Lemma 3.3. Let $\varphi \in \text{INEX}_L$, $\vec{t} \in T_L$ a k -tuple and let \vec{u} be a k -tuple of variables that are not in $\text{Vr}(\vec{t})$. Let $X' := \{s[s(\vec{t})/\vec{u}] \mid s \in X\}$. Now we have $X(\vec{t}) = X'(\vec{u})$ and the following holds: $\mathcal{M} \models_X [\vec{t} \triangleright \vec{u}]\varphi$ iff $\mathcal{M} \models_{X'} \varphi$.

Existential inclusion and exclusion quantifiers

We begin by defining semantics for existential inclusion and exclusion quantifiers $(\exists x \subseteq y)$ and $(\exists x | y)$. We take here a slightly more general approach by allowing the variable y to be any L -term t . A natural reading for *existential inclusion quantifier* $(\exists x \subseteq t)$ is that “there exists an x within the values of t ”.

This kind of truth condition can be achieved simply by modifying the standard truth condition of existential quantifier in such a way that the values given by the choice function F are restricted to the values of t in a team X . We then obtain the following truth condition:

$$\mathcal{M} \models_X (\exists x \subseteq t) \varphi \text{ iff there is } F : X \rightarrow \mathcal{P}^*(X(t)) \text{ s.t. } \mathcal{M} \models_{X[F/x]} \varphi.$$

Another natural language interpretation for quantifier $(\exists x \subseteq t)$ is that the values given for x must be *possible* for the term t . From the perspective of semantic games we may say that *verifying* player's allowed moves are restricted on the set $X(t)$ instead of the whole universe of a model.³

Similarly we read *existential exclusion quantifier* $(\exists x | t)$ as “there exists an x outside the values of t ”. To achieve this, we simply restrict values given by the choice function F to the complement, $\overline{X(t)} = M \setminus X(t)$, of $X(t)$:

$$\mathcal{M} \models_X (\exists x | t) \varphi \text{ iff there is } F : X \rightarrow \mathcal{P}^*(\overline{X(t)}) \text{ s.t. } \mathcal{M} \models_{X[F/x]} \varphi.$$

This kind of quantification dually must give such values for x that are *not possible* for t . Or in a semantic game we can say that the values in the set $X(t)$ are “banned” from the verifier when (s)he chooses a value for x .

Next we define these operators, as abbreviations, by using inclusion and exclusion atoms. We want their truth conditions to be as described above, but we give the definitions in a more general form by allowing the quantification of tuples instead of just single variables.

Definition 3.4. Let $\varphi \in \text{INEX}_L$, $\vec{t} \in T_L$ a k -tuple and let \vec{x}, \vec{u} be k -tuples of variables s.t. the variables in \vec{u} are not in $\text{Vr}(\vec{t})$. We use the following notations:

$$\begin{aligned} (\exists \vec{x} \subseteq \vec{t}) \varphi &:= [\vec{t} \triangleright \vec{u}] \exists \vec{x} (\vec{x} \subseteq \vec{u} \wedge \varphi) \\ (\exists \vec{x} | \vec{t}) \varphi &:= [\vec{t} \triangleright \vec{u}] \exists \vec{x} (\vec{x} | \vec{u} \wedge \varphi). \end{aligned}$$

Note that the lengths of quantified tuples match the arities of atoms, i.e. if $\varphi \in \text{INEX}_L[k]$ and \vec{x}, \vec{t} are k -tuples, then $(\exists \vec{x} \subseteq \vec{t}) \varphi, (\exists \vec{x} | \vec{t}) \varphi \in \text{INEX}[k]$. Since only one type of atom is needed for each quantifier, $(\exists \vec{x} \subseteq \vec{t}) \varphi \in \text{INC}_L[k]$ when $\varphi \in \text{INC}_L[k]$ and $(\exists \vec{x} | \vec{t}) \varphi \in \text{EXC}[k]$ when $\varphi \in \text{EXC}_L[k]$. Also note that $\text{Fr}((\exists \vec{x} \subseteq \vec{t}) \varphi) = (\text{Fr}(\varphi) \setminus \{x\}) \cup \text{Vr}(\vec{t}) = \text{Fr}((\exists \vec{x} | \vec{t}) \varphi)$.

The next proposition presents the truth conditions given by Definition 3.4.

³Note that the setting here is quite different than in IF-logic (or dependence logic). In IF-logic the verifying player is allowed to choose any values, but values for certain variables are “hidden” from him/her when making the choice. Here the player may see the values of all variables, but only certain values are admissible to be chosen. In the former case the *domain* of the strategy function is restricted and in the latter case only its *range* is restricted.

In the proof of the next proposition, and from now on, we will write $\text{ran}(F) := \{F(s) \mid s \in X\}$ for any function F that is defined in some team X .

Proposition 3.4. *With the same assumption as in Definition 3.4, we obtain the following truth conditions:*

- a) $\mathcal{M} \models_X (\exists \vec{x} \subseteq \vec{t}) \varphi$ iff there is $\mathcal{F} : X \rightarrow \mathcal{P}^*(X(\vec{t}))$ s.t. $\mathcal{M} \models_{X[\mathcal{F}/\vec{x}]} \varphi$.
- b) $\mathcal{M} \models_X (\exists \vec{x} \mid \vec{t}) \varphi$ iff there is $\mathcal{F} : X \rightarrow \mathcal{P}^*(\overline{X(\vec{t})})$ s.t. $\mathcal{M} \models_{X[\mathcal{F}/\vec{x}]} \varphi$.

Proof. By locality we may assume that the variables in \vec{u} are not in $\text{dom}(X)$. We write $X' := \{s[s(\vec{t})/u] \mid s \in X\}$.

a) Suppose $\mathcal{M} \models_X (\exists \vec{x} \subseteq \vec{t}) \varphi$. By Lemma 3.3, $\mathcal{M} \models_{X'} \exists \vec{x} (\vec{x} \subseteq \vec{u} \wedge \varphi)$. Thus there is $\mathcal{F}' : X' \rightarrow \mathcal{P}^*(M^k)$ s.t. $\mathcal{M} \models_{X'[\mathcal{F}'/\vec{x}]} \vec{x} \subseteq \vec{u} \wedge \varphi$. Let $\mathcal{F} : X \rightarrow \mathcal{P}^*(M)$ s.t. $s \mapsto \mathcal{F}'(s[s(\vec{t})/\vec{u}])$. Now by locality $\mathcal{M} \models_{X[\mathcal{F}/\vec{x}]} \varphi$. We still need to show that $\text{ran}(\mathcal{F}) \subseteq \mathcal{P}^*(X(\vec{t}))$. Since $X'(\vec{u}) = X(\vec{t})$, this amounts to showing that $\text{ran}(\mathcal{F}') \subseteq \mathcal{P}^*(X'(\vec{u}))$. But this is easy to see since $\mathcal{M} \models_{X'[\mathcal{F}'/\vec{x}]} \vec{x} \subseteq \vec{u}$.

Suppose then that there is $\mathcal{F} : X \rightarrow \mathcal{P}^*(X(\vec{t}))$ s.t. $\mathcal{M} \models_{X[\mathcal{F}/\vec{x}]} \varphi$. Let $\mathcal{F}' : X' \rightarrow \mathcal{P}^*(M)$ s.t. $s \mapsto \mathcal{F}(s \upharpoonright \text{dom}(X))$. Since $\text{ran}(\mathcal{F}) \subseteq \mathcal{P}^*(X(\vec{t}))$, we also have $\text{ran}(\mathcal{F}') \subseteq \mathcal{P}^*(X'(\vec{u}))$. Thus it is easy to show that $\mathcal{M} \models_{X'[\mathcal{F}'/\vec{x}]} \vec{x} \subseteq \vec{u}$. By locality $\mathcal{M} \models_{X'[\mathcal{F}'/\vec{x}]} \varphi$ and thus by Lemma 3.3 we have $\mathcal{M} \models_X (\exists \vec{x} \subseteq \vec{t}) \varphi$.

The case b) is proven similarly. For a more detailed proof see [17]. \square

Remark. When defining these quantifiers, we did not want to put any restrictions on tuples \vec{x}, \vec{t} and thus, in particular, we also allow the variables in \vec{x} to occur in \vec{t} . This is why we need to use the storing operator, since the values of \vec{t} in a team might change after the quantification of \vec{x} .

If we would drop the storing operator from Definition 3.4, then the choice function \mathcal{F} would be required to choose values within (\subseteq) or outside (\mid) the set $X[\mathcal{F}/\vec{x}](\vec{t})$ instead of the set $X(\vec{t})$. Hence the values in the team *after* the quantification would restrict the range of the choice function that is used for the quantification. This would lead to a very unnatural truth condition.

Since quantifications may change the values of terms in a team, several identical consecutive existential inclusion/exclusion quantifications can change the meaning of a formula, as seen by the following example.

Example 3.1. Let $c \in L$ be a constant symbol and $f \in L$ a unary function symbol. Let

$$\begin{aligned} \varphi &:= \exists x (\exists x \subseteq fx)(x = c) \\ \psi &:= \exists x (\exists x \subseteq fx)(\exists x \subseteq fx)(x = c). \end{aligned}$$

The sentences φ and ψ are not logically equivalent since $\varphi \equiv \exists x (fx = c)$, but $\psi \equiv \exists x (ffx = c)$.

The following example presents a property that is not FO-definable, but can be expressed with a sentence containing a single existential inclusion quantifier. A similar example was presented originally for inclusion logic in [6].

Example 3.2. A directed finite graph $\mathcal{G} = (V, E)$ contains a cycle if and only if the following holds:

$$\mathcal{G} \models \exists x (\exists y \subseteq x) Exy.$$

Universal inclusion and exclusion quantifiers

Next we define semantics for universal inclusion and exclusion quantifiers $(\forall x \subseteq y)$ and $(\forall x | y)$. Again we allow y to be any L -term t and first consider the semantics for these operators from an intuitive perspective. For *universal inclusion quantifier* $(\forall x \subseteq t)$ a natural reading would be: “for all x within the values of t ”. This restricted universal quantification is done simply by quantifying x over the set $X(t)$ instead of the whole universe M :

$$\mathcal{M} \models_X (\forall x \subseteq t) \varphi \quad \text{iff} \quad \mathcal{M} \models_{X[A/x]} \varphi, \text{ where } A = X(t).$$

A dualistic reading for *universal exclusion quantifier* $(\forall x | t)$ is “for all x outside the values of t ”. This is achieved by quantifying x over the *complement* of $X(t)$:

$$\mathcal{M} \models_X (\forall x | t) \varphi \quad \text{iff} \quad \mathcal{M} \models_{X[A/x]} \varphi, \text{ where } A = \overline{X(t)}.$$

As with existential inclusion and exclusion quantifiers, we can observe the semantics above from a game-theoretic perspective by restricting the allowed moves of the players. This time, when choosing values for x , the *falsifying* player may only choose the values of t in the case of inclusion, and the values of t are forbidden from him/her in the case of exclusion.

Next we define these operators as abbreviations in INEX, aiming for the truth conditions as described above. Again we give the definitions in a more general form by using tuples instead of just single variables. The definitions here turn out to be much more complicated than the ones for existential quantifiers.

Definition 3.5. Let $\varphi \in \text{INEX}_L$, $\vec{t} \in T_L$ a k -tuple, \vec{x} a k -tuple of variables and $\vec{u}, \vec{y}, \vec{z}$ k -tuples of fresh variables. We use the following notations:

$$\begin{aligned} (\forall \vec{x} \subseteq \vec{t}) \varphi &:= [\vec{t} \triangleright \vec{u}] \Big(\forall \vec{x} (\vec{x} \subseteq \vec{u} \wedge \varphi) \\ &\quad \sqcup \forall \vec{x} (\exists \vec{y} \subseteq \vec{u}) (\exists \vec{z} | \vec{u}) \Big((\vec{x} = \vec{y} \wedge \varphi) \vee \vec{x} = \vec{z} \Big) \Big) \\ (\forall \vec{x} | \vec{t}) \varphi &:= [\vec{t} \triangleright \vec{u}] \Big(\forall \vec{x} (\vec{x} \subseteq \vec{u}) \\ &\quad \sqcup \forall \vec{x} (\exists \vec{y} \subseteq \vec{u}) (\exists \vec{z} | \vec{u}) \Big(\vec{x} = \vec{y} \vee (\vec{x} = \vec{z} \wedge \varphi) \Big) \Big). \end{aligned}$$

Also here the arities match: if $\varphi \in \text{INEX}_L[k]$ and \vec{x}, \vec{t} are k -tuples, then $(\forall \vec{x} \subseteq \vec{t}) \varphi, (\forall \vec{x} | \vec{t}) \varphi \in \text{INEX}[k]$. But since we need both inclusion and exclusion atoms for both of these definitions, neither of these quantifiers can be

defined in this way in just INC or EXC. It is thus natural to ask whether we could give these definitions in a way that only one type of atoms would be used for each definition – we will get back to this question in the next subsection.

Proposition 3.5. *With the same assumptions as in Definition 3.5, we obtain the following truth conditions:*

- a) $\mathcal{M} \models_X (\forall \vec{x} \subseteq \vec{t}) \varphi$ iff $\mathcal{M} \models_{X[A/\vec{x}]} \varphi$, where $A = X(\vec{t})$.
- b) $\mathcal{M} \models_X (\forall \vec{x} \mid \vec{t}) \varphi$ iff $\mathcal{M} \models_{X[\overline{A}/\vec{x}]} \varphi$, where $A = \overline{X(\vec{t})}$.

The idea of the proofs for these truth conditions is that the trivial case when $X(\vec{t}) = M^k$ is dealt on left side of the intuitionistic disjunction. If $X(\vec{t}) \neq M^k$, we first universally quantify \vec{x} and then split the resulting team into subteams Y, Y' such that $Y = X[X(\vec{t})/\vec{x}]$ and $Y' = X[\overline{X(\vec{t})}/\vec{x}]$. Then we just say that the formula φ holds on the desired side.

We first prove the following claim which shows how we can force the team $X[M^k/\vec{x}]$ to be split into the subteams $X[X(\vec{t})/\vec{x}]$ and $X[\overline{X(\vec{t})}/\vec{x}]$.

Claim 1. *Let $\psi, \theta \in \text{INEX}_L$, $\vec{t} \in T_L$, let \vec{x} be a k -tuple of variables, and let \vec{y}, \vec{z} be k -tuples of fresh variables. We additionally assume here that $X(\vec{t}) \neq M^k$ and that the variables in the tuple \vec{x} are not in $\text{Vr}(\vec{t})$. Let*

$$\xi := \forall \vec{x} (\exists \vec{y} \subseteq \vec{t}) (\exists \vec{z} \mid \vec{t}) ((\vec{x} = \vec{y} \wedge \psi) \vee (\vec{x} = \vec{z} \wedge \theta)).$$

Now we have: $\mathcal{M} \models_X \xi$ iff $\mathcal{M} \models_{X[X(\vec{t})/\vec{x}]} \psi$ and $\mathcal{M} \models_{X[\overline{X(\vec{t})}/\vec{x}]} \theta$.

Proof. By locality we may assume that $\text{dom}(X) = \text{Fr}(\xi)$. Suppose first $\mathcal{M} \models_X \xi$. Thus there are $\mathcal{F}_1 : X_1 \rightarrow \mathcal{P}^*(X_1(\vec{t}))$ and $\mathcal{F}_2 : X_2 \rightarrow \mathcal{P}^*(X_2(\vec{t}))$ such that $\mathcal{M} \models_{X_3} (\vec{x} = \vec{y} \wedge \psi) \vee (\vec{x} = \vec{z} \wedge \theta)$, where $X_1 := X[M^k/\vec{x}]$, $X_2 := X_1[\mathcal{F}_1/\vec{y}]$ and $X_3 := X_2[\mathcal{F}_2/\vec{z}]$. Furthermore there are $Y, Y' \subseteq X_3$ s.t. $Y \cup Y' = X_3$, $\mathcal{M} \models_Y \vec{x} = \vec{y} \wedge \psi$ and $\mathcal{M} \models_{Y'} \vec{x} = \vec{z} \wedge \theta$.

Since $\mathcal{M} \models_Y \psi$, $\mathcal{M} \models_{Y'} \theta$ and $\vec{y}, \vec{z} \notin \text{dom}(X_1)$, by locality it is sufficient to show that $X[X(\vec{t})/\vec{x}] = Y \upharpoonright \text{dom}(X_1)$ and $X[\overline{X(\vec{t})}/\vec{x}] = Y' \upharpoonright \text{dom}(X_1)$. But this is quite easy to see since $\mathcal{M} \models_Y \vec{x} = \vec{y}$, $\mathcal{M} \models_{Y'} \vec{x} = \vec{z}$ and the values of \vec{y} were chosen within the values of \vec{t} and the values of \vec{z} outside the values of \vec{t} .

Suppose then that $\mathcal{M} \models_{X[X(\vec{t})/\vec{x}]} \psi$ and $\mathcal{M} \models_{X[\overline{X(\vec{t})}/\vec{x}]} \theta$. We may assume that $X \neq \emptyset$, because otherwise the claim would hold trivially. Since now $X(\vec{t}) \neq \emptyset$ and by assumption $X(\vec{t}) \neq M^k$, there is $\vec{a}^* \in X(\vec{t})$ and $\vec{b}^* \in \overline{X(\vec{t})}$.

Let $X_1 := X[M^k/\vec{x}]$. We define $\mathcal{F}_1 : X_1 \rightarrow \mathcal{P}^*(M^k)$ s.t. it maps s to $\{s(\vec{x})\}$ if $s(\vec{x}) \in X_1(\vec{t})$ and else maps s to $\{\vec{a}^*\}$. Let $X_2 := X_1[\mathcal{F}_1/\vec{y}]$. We then define $\mathcal{F}_2 : X_2 \rightarrow \mathcal{P}^*(M^k)$ s.t. it maps s to $\{s(\vec{x})\}$ if $s(\vec{x}) \in \overline{X_2(\vec{t})}$ and else maps s to $\{\vec{b}^*\}$. Let $X_3 := X_2[\mathcal{F}_2/\vec{z}]$. Clearly now $\text{ran}(\mathcal{F}_1) \subseteq \mathcal{P}^*(X_1(\vec{t}))$ and $\text{ran}(\mathcal{F}_2) \subseteq \mathcal{P}^*(X_2(\vec{t}))$.

Let $Y := \{s \in X_3 \mid s(\vec{x}) \in X_3(\vec{t})\}$ and $Y' := \{s \in X_3 \mid s(\vec{x}) \in \overline{X_3(\vec{t})}\}$. Now clearly $Y \cup Y' = X_3$, $\mathcal{M} \models_Y \vec{x} = \vec{y}$ and $\mathcal{M} \models_{Y'} \vec{x} = \vec{z}$. Also it is quite easy to see that $X[X(\vec{t})/\vec{x}] = Y \upharpoonright \text{dom}(X_1)$ and $X[\overline{X(\vec{t})}/\vec{x}] = Y' \upharpoonright \text{dom}(X_1)$, whence by locality $\mathcal{M} \models_Y \psi$ and $\mathcal{M} \models_{Y'} \theta$. Hence $\mathcal{M} \models_{X_3} (\vec{x} = \vec{y} \wedge \psi) \vee (\vec{x} = \vec{z} \wedge \theta)$, and furthermore we have $\mathcal{M} \models_X \xi$. For a more detailed proof, see [17]. \square

By using Claim 1, we can easily prove the truth conditions for universal inclusion and exclusion quantifiers (Proposition 3.5). We only need to consider the use of storing operator and the special case when $X(\vec{t}) = M^k$. When using the storing operator, we may drop the extra assumption that $\text{Vr}(\vec{x}) \cap \text{Vr}(\vec{t}) = \emptyset$. When $X(\vec{t}) = M^k$ the universal inclusion quantifier $(\forall \vec{x} \subseteq \vec{t})$ becomes the normal universal quantifier $\forall \vec{x}$ and the universal exclusion quantifier $(\forall \vec{x} \mid \vec{t})$ becomes trivially true. For the rest of the proof we apply Claim 1 with $\psi := \varphi$ and $\theta := (\vec{x} = \vec{x})$ for $(\forall \vec{x} \subseteq \vec{t}) \varphi$, and $\psi := (\vec{x} = \vec{x})$ and $\theta := \varphi$ for $(\forall \vec{x} \mid \vec{t}) \varphi$. For all technical details, see [17].

A natural idea for the truth definition for universal inclusion quantification $(\forall \vec{x} \subseteq \vec{y})$ is “ $\forall \vec{x} \in M^k : (\vec{x} \subseteq \vec{y} \Rightarrow \varphi)$ ”. This intuition would give us the following definition: $(\forall \vec{x} \subseteq \vec{y}) \varphi := \forall \vec{x} (\vec{x} \mid \vec{y} \vee \varphi)$. However, this simple idea does not work for two reasons. Firstly, there might be too many values chosen for \vec{x} on the right side of the disjunction, which can be a problem since INEX is not closed downwards. Secondly, the exclusion atom is evaluated *after* splitting the team and thus some of the original values for \vec{y} might be lost. This general problem regarding the “loss of information” when evaluating disjunctions will be discussed more in the Subsection 3.4, where we define term value preserving disjunction.

3.3. Analyzing the properties of inclusion and exclusion quantifiers

In the previous subsections we showed that inclusion and exclusion quantifiers can be expressed with inclusion and exclusion atoms, and thus we were able to define them as abbreviations in INEX. In this subsection we take a reverse perspective by considering them as basic operations to be added to FO and examining the expressive power of the resulting logics. The following observation shows that we can define inclusion and exclusion atoms with existential inclusion and exclusion quantifiers $(\exists \vec{x} \subseteq \vec{t})$ and $(\exists \vec{x} \mid \vec{t})$.

Observation 3.1. Let \vec{t}_1, \vec{t}_2 be k -tuples of L -terms and let \vec{x} be a k -tuple of fresh variables. Now it holds that:

$$\begin{aligned} \mathcal{M} \models_X \vec{t}_1 \subseteq \vec{t}_2 & \text{ iff } \mathcal{M} \models_X (\exists \vec{x} \subseteq \vec{t}_2) (\vec{x} = \vec{t}_1). \\ \mathcal{M} \models_X \vec{t}_1 \mid \vec{t}_2 & \text{ iff } \mathcal{M} \models_X (\exists \vec{x} \mid \vec{t}_2) (\vec{x} = \vec{t}_1), \end{aligned}$$

We explain briefly why these equivalences hold. We first notice that for any function $\mathcal{F} : X \rightarrow \mathcal{P}^*(M^k)$ the following holds:

$$\mathcal{M} \models_{X[\mathcal{F}/\vec{x}]} \vec{x} = \vec{t}_1 \text{ iff } \mathcal{F}(s) = \{s(\vec{t}_1)\} \text{ for each } s \in X. \quad (\star)$$

It is easy to see that if \mathcal{F} is a function which satisfies the (both) sides of (\star) , then we have: $\text{ran}(\mathcal{F}) \subseteq \mathcal{P}^*(X(\vec{t}_2))$ if and only if $\mathcal{M} \models_X \vec{t}_1 \subseteq \vec{t}_2$. The first equivalence follows from this. The second one is also clear since $\text{ran}(\mathcal{F}) \subseteq \mathcal{P}^*(\overline{X(\vec{t}_2)})$ iff $\mathcal{M} \models_X \vec{t}_1 \mid \vec{t}_2$, for any \mathcal{F} which satisfies the both sides of (\star) .

Recall that, in Definition 3.4, we were able to define the quantifier $(\exists \vec{x} \subseteq \vec{t})$ with inclusion atom and the quantifier $(\exists \vec{x} \mid \vec{t})$ with exclusion atom. Hence, by the previous observation, if we extend FO with quantifiers $(\exists \vec{x} \subseteq \vec{t})$ or $(\exists \vec{x} \mid \vec{t})$, we obtain equivalent logics with INC and EXC, respectively. We call these logics *inclusion and exclusion friendly logics* due their similarity with IF-logic. By using the both of these quantifiers, we obtain *inclusion-exclusion friendly logic* that is equivalent with INEX.

Also note that the arities of these operations match, since the use of existential inclusion (exclusion) quantifiers for k -tuples corresponds to the use of k -ary inclusion (exclusion) atoms. Hence the use of existential inclusion and exclusion quantifiers for *single* first order variables corresponds to the use of *unary* inclusion and exclusion atoms, and thus, by extending FO with either/both of them, we obtain logics equivalent to INC[1], EXC[1] and INEX[1].

After the Observation 3.1 it is natural to ask whether we can define inclusion and exclusion atoms alternatively by using universal inclusion and exclusion quantifiers $(\forall \vec{x} \subseteq \vec{t})$ and $(\forall \vec{x} \mid \vec{t})$. This can also be done, however, this time inclusion atom is defined with universal *exclusion* quantifier and exclusion atom is defined with universal *inclusion* quantifier.

Observation 3.2. Let \vec{t}_1, \vec{t}_2 be k -tuples of L -terms and let \vec{x} be a k -tuple of fresh variables. Now the following equivalences hold:

$$\begin{aligned} \mathcal{M} \models_X \vec{t}_1 \subseteq \vec{t}_2 & \text{ iff } \mathcal{M} \models_X (\forall \vec{x} \mid \vec{t}_2)(\vec{x} \neq \vec{t}_1) \\ \mathcal{M} \models_X \vec{t}_1 \mid \vec{t}_2 & \text{ iff } \mathcal{M} \models_X (\forall \vec{x} \subseteq \vec{t}_2)(\vec{x} \neq \vec{t}_1), \end{aligned}$$

We prove the first equivalence by contraposition: Suppose that $\mathcal{M} \not\models_X \vec{t}_1 \subseteq \vec{t}_2$, i.e. there is $s \in X$ such that $s(\vec{t}_1) \notin X(\vec{t}_2)$. Let $r := s[s(\vec{t}_1)/\vec{x}]$, whence $r \in X[\overline{X(\vec{t}_2)}/\vec{x}]$. Now $r(\vec{x}) = s(\vec{t}_1) = r(\vec{t}_1)$ and thus $\mathcal{M} \not\models_X (\forall \vec{x} \mid \vec{t}_2)(\vec{x} \neq \vec{t}_1)$.

For the other direction suppose that $\mathcal{M} \not\models_X (\forall \vec{x} \mid \vec{t}_2)(\vec{x} \neq \vec{t}_1)$, whence there is $r \in X[\overline{X(\vec{t}_2)}/\vec{x}]$ such that $r(\vec{x}) = r(\vec{t}_1)$. Now there is $s \in X$ and $\vec{a} \in \overline{X(\vec{t}_2)}$ such that $r = s[\vec{a}/\vec{x}]$. But since $s(\vec{t}_1) = r(\vec{t}_1) = r(\vec{x}) = \vec{a} \notin X(\vec{t}_2)$, we have $\mathcal{M} \not\models_X \vec{t}_1 \subseteq \vec{t}_2$. The second equivalence can be proven by a similar reasoning.

When we combine the equivalences above with the respective equivalences in Observation 3.1, we obtain the following correspondence.

$$\begin{aligned}(\exists \vec{x} \subseteq \vec{t}_2)(\vec{x} = \vec{t}_1) &\equiv (\forall \vec{x} \mid \vec{t}_2)(\vec{x} \neq \vec{t}_1) \\(\exists \vec{x} \mid \vec{t}_2)(\vec{x} = \vec{t}_1) &\equiv (\forall \vec{x} \subseteq \vec{t}_2)(\vec{x} \neq \vec{t}_1).\end{aligned}$$

Here we have an interesting duality between the inclusion and exclusion quantifiers. This leads to a natural question whether existential inclusion quantifier $(\exists \vec{x} \subseteq \vec{t})$ has the same expressive power as universal exclusion quantifier $(\forall \vec{x} \mid \vec{t})$, and the whether the same holds for the quantifiers $(\exists \vec{x} \mid \vec{t})$ and $(\forall \vec{x} \subseteq \vec{t})$. We approach this question by first comparing universal inclusion/exclusion quantifiers with INC and EXC.

In Definition 3.5 we defined universal inclusion and exclusion quantifiers in INEX by using both inclusion and exclusion atoms. We examine next whether either of them could be defined by using only one type of these atoms. For the next observation, recall that EXC is closed downwards and INC under unions.

Observation 3.3. Let $\mathcal{M} = (\mathcal{I}, M)$ be an L -model s.t. $M = \{0, 1, 2\}$, and let $X_1 = \{s_{01}\}$ and $X_2 = \{s_{10}\}$, where $s_{01}(x) = 0 = s_{10}(y)$ and $s_{01}(y) = 1 = s_{10}(x)$.

(A) We first show that universal inclusion quantifier is not closed under unions. For this, let $\varphi := (\forall z \subseteq x)(y \neq z)$. We consider the following teams

$$\begin{aligned}Y_1 &:= X_1[X_1(x)/z] = X_1[\{0\}/z] = \{s_{01}[0/z]\} \\Y_2 &:= X_2[X_2(x)/z] = X_2[\{1\}/z] = \{s_{10}[1/z]\} \\Y_3 &:= (X_1 \cup X_2)[(X_1 \cup X_2)(x)/z] = (X_1 \cup X_2)[\{0, 1\}/z] \\&= \{s_{01}[0/z], s_{01}[1/z], s_{10}[0/z], s_{10}[1/z]\}.\end{aligned}$$

Now we have $\mathcal{M} \models_{Y_1} y \neq z$ and $\mathcal{M} \models_{Y_2} y \neq z$, but $\mathcal{M} \not\models_{Y_3} y \neq z$. Hence $\mathcal{M} \models_{X_1} \varphi$ and $\mathcal{M} \models_{X_2} \varphi$, but $\mathcal{M} \not\models_{X_1 \cup X_2} \varphi$.

(B) Next, we show that universal exclusion quantifier is not closed under unions. Let $\psi := (\forall z \mid x)(y \subseteq z)$. Note that, by Observation 3.2, $y \subseteq z$ can be expressed with universal exclusion quantifier ($\psi \equiv (\forall z \mid x)(\forall w \mid z)(w \neq y)$). Let

$$\begin{aligned}Z_1 &:= X_1[\overline{X_1(x)}/z] = X_1[\overline{\{0\}}/z] = X_1[\{1, 2\}/z] = \{s_{01}[1/z], s_{01}[2/z]\} \\Z_2 &:= X_2[\overline{X_2(x)}/z] = X_2[\overline{\{1\}}/z] = X_2[\{0, 2\}/z] = \{s_{10}[0/z], s_{10}[2/z]\} \\Z_3 &:= (X_1 \cup X_2)[\overline{(X_1 \cup X_2)(x)}/z] = (X_1 \cup X_2)[\overline{\{0, 1\}}/z] \\&= (X_1 \cup X_2)[\{2\}/z] = \{s_{01}[2/z], s_{10}[2/z]\}.\end{aligned}$$

Since $Z_1(y) = \{1\} \subseteq \{1, 2\} = Z_1(z)$ and $Z_2(y) = \{0\} \subseteq \{0, 2\} = Z_2(z)$, we have $\mathcal{M} \models_{Z_1} y \subseteq z$ and $\mathcal{M} \models_{Z_2} y \subseteq z$. But because $Z_3(y) = \{0, 1\} \not\subseteq \{2\} = Z_3(z)$,

we have $\mathcal{M} \not\models_{Z_3} y \subseteq z$. Hence $\mathcal{M} \models_{X_1} \psi$ and $\mathcal{M} \models_{X_2} \psi$, but $\mathcal{M} \not\models_{X_1 \cup X_2} \psi$.

(C) Finally, we show that universal exclusion quantifier is not closed downwards either. Let $\theta := (\forall z | x)(y \neq z)$ and let Z_1, Z_3 be as above. Now $\mathcal{M} \models_{Z_3} y \neq z$, but $\mathcal{M} \not\models_{Z_1} y \neq z$. Hence $\mathcal{M} \models_{X_1 \cup X_2} \theta$, but $\mathcal{M} \not\models_{X_1} \theta$; even though $X_1 \subseteq X_1 \cup X_2$.

By this observation, universal exclusion quantifier cannot be defined in EXC and *neither* universal inclusion nor exclusion quantifier can be defined in INC. But there is still a possibility that universal inclusion quantifier could be defined in EXC. It turns out that this can indeed be done, but we must give its definition in a form that would not work properly in INEX. To make distinction with the earlier definition, we denote this quantifier $(\forall \vec{x} \subseteq^e \vec{t})$, where “e” stands for “exclusion”, as this operator is defined for exclusion logic only.

Definition 3.6. Let $\varphi \in \text{EXC}_L$, $\vec{t} \in T_L$ a k -tuple, \vec{x} a k -tuple of variables and \vec{u}, \vec{y} k -tuples of fresh variables. We use the following notation:

$$(\forall \vec{x} \subseteq^e \vec{t}) \varphi := \forall \vec{x} \varphi \sqcup [\vec{t} \triangleright \vec{u}] \forall \vec{x} (\exists \vec{y} | \vec{u})(\vec{y} = \vec{x} \vee \varphi).$$

Since intuitionistic disjunction can be defined with unary exclusion atoms we have $(\forall \vec{x} \subseteq^e \vec{t}) \varphi \in \text{EXC}_L[k]$ when $\varphi \in \text{EXC}_L[k]$ (for any $k \geq 1$).

Proposition 3.6. *With the same assumptions as in Definition 3.6, we obtain the following truth condition: $\mathcal{M} \models_X (\forall \vec{x} \subseteq^e \vec{t}) \varphi$ iff $\mathcal{M} \models_{X[X(\vec{t})/\vec{x}]} \varphi$.*

The proof for this truth condition can be done using a similar reasoning as for the standard inclusion quantifier $(\forall \vec{x} \subseteq \vec{t})$. Note that since EXC is closed downwards, the truth of $\forall \vec{x} \varphi$ in X implies that $\mathcal{M} \models_{X[X(\vec{t})/\vec{x}]} \varphi$. Also, again by downwards closure, when the team $X[M^k/\vec{x}]$ is split into two subteams, if $\mathcal{M} \models_Y \varphi$ for some team Y for which $X[X(\vec{t})/\vec{x}] \subseteq Y$, then also $\mathcal{M} \models_{X[X(\vec{t})/\vec{x}]} \varphi$. For a full proof with all technical details, see [17].

For proving the truth condition for $(\forall \vec{x} \subseteq^e \vec{t})$ we had to use the assumption of downwards closure which does not hold for INEX. Moreover, the claim of Proposition 3.6 is not necessarily true when $\varphi \in \text{INEX}_L$ since, for example, if $\varphi := \forall x (x \subseteq y)$ and $X(z) \neq M$, then $\mathcal{M} \models_X (\forall y \subseteq^e z) \varphi$, but $\mathcal{M} \not\models_X (\forall y \subseteq z) \varphi$.

By the observation above, we see that definability of these quantifiers, as well as many other operators for team semantics, is “case sensitive”. That is, if a certain operator O is definable in a logic \mathcal{L} and \mathcal{L}' is an extension of \mathcal{L} , then the operator O may have to be defined differently in \mathcal{L}' . Note that atoms in team semantics are more regular in this sense, since if a certain atom A is definable in a logic \mathcal{L} , then A can be defined in all of the extensions of \mathcal{L} identically as it is defined in \mathcal{L} .

Since we were able to define universal inclusion quantifier $(\forall \vec{x} \subseteq \vec{t})$ in EXC, it would have been natural to predict that universal exclusion quantifier $(\forall \vec{x} | \vec{t})$

is dually definable in INC. However, this is impossible since this operator is not closed under unions as shown in Observation 3.3. Here we have an interesting piece of asymmetry between the inclusion and exclusion operators.

In this subsection we were able to show that *existential* inclusion and exclusion quantifiers are very closely related to inclusion and exclusion atoms. However, perhaps a bit surprisingly, with *universal* inclusion and exclusion quantifiers, this relationship becomes more complicated. One interesting question, that is still open, is the exact expressive power of universal exclusion quantifier. For now, we only know that when \vec{x} and \vec{t} are k -ary, then $(\forall \vec{x} | \vec{t})$ is (strictly) stronger than k -ary inclusion atom. However, it is possible that this difference would disappear on the level of sentences – that is, FO extended with $(\forall \vec{x} | \vec{t})$ (where \vec{x}, \vec{t} are k -ary) would become equivalent with INC[k] when we only consider sentences. We leave this question open for further research.

3.4. Term value preserving disjunction

When evaluating disjunctions, the team is split and usually some information is lost about the values of terms in the original team. Often this is desirable, since we want to shrink or distribute the values of certain variables by giving conditions on the disjuncts.

However, sometimes we want that the values of certain terms (or tuples of terms) are preserved on both sides after the evaluation of the disjunction. This is desirable especially when we are using variables to carry information about sets (or tuples of variables to carry information about relations). This method will be crucial in the proof of Theorem 4.5 later in this paper.

For this purpose we introduce *term value preserving disjunction*. It can be defined by using constancy atoms, intuitionistic disjunctions and inclusion atoms of the same arity as the lengths of the tuples whose values we want to preserve. Thus, with this operator, the values of single terms can be preserved in INEX[1] and the values of k -tuples of terms can be preserved in INEX[k].

Definition 3.7. Let $\vec{t}_1, \dots, \vec{t}_n$ be k -tuples of L -terms, $\varphi, \psi \in \text{INEX}_L$ and c_l, c_r, y fresh variables. We define

$$\begin{aligned} \varphi_{\vec{t}_1, \dots, \vec{t}_n} \vee \psi &:= (\varphi \sqcup \psi) \sqcup \exists c_l \exists c_r \left(= (c_l) \wedge = (c_r) \wedge c_l \neq c_r \right. \\ &\quad \left. \wedge \exists y \left(((y = c_l \wedge \varphi) \vee (y = c_r \wedge \psi)) \wedge \bigwedge_{i \leq n} (\theta_i \wedge \theta'_i) \right) \right), \\ \theta_i &:= \exists \vec{z}_1 \exists \vec{z}_2 \left(((y = c_l \wedge \vec{z}_1 = \vec{t}_i \wedge \vec{z}_2 = \vec{c}_1) \right. \\ &\quad \left. \vee (y = c_r \wedge \vec{z}_1 = \vec{c}_1 \wedge \vec{z}_2 = \vec{t}_i)) \wedge \vec{t}_i \subseteq \vec{z}_1 \wedge \vec{t}_i \subseteq \vec{z}_2 \right) \\ \theta'_i &:= \exists \vec{z}_1 \exists \vec{z}_2 \left(((y = c_l \wedge \vec{z}_1 = \vec{t}_i \wedge \vec{z}_2 = \vec{c}_2) \right. \\ &\quad \left. \vee (y = c_r \wedge \vec{z}_1 = \vec{c}_2 \wedge \vec{z}_2 = \vec{t}_i)) \wedge \vec{t}_i \subseteq \vec{z}_1 \wedge \vec{t}_i \subseteq \vec{z}_2 \right), \end{aligned}$$

where $\vec{z}_1, \vec{z}_2, \vec{c}_1, \vec{c}_2$ are k -tuples of variables such that the tuples \vec{z}_1, \vec{z}_2 consist of fresh variables, and \vec{c}_1, \vec{c}_2 are defined as $\vec{c}_1 := c_l \dots c_l$ and $\vec{c}_2 := c_r \dots c_r$.

The next proposition gives the truth condition for this operator. Note that this truth condition is the same as for the normal disjunction with an extra condition that the values for the tuples $\vec{t}_1, \dots, \vec{t}_n$ must be preserved on both sides after splitting the team (supposing that the split is nontrivial).

Proposition 3.7. *With the same assumptions as in Definition 3.7, we obtain the following truth condition:*

$$\begin{aligned} \mathcal{M} \models_X \varphi \vee_{\vec{t}_1, \dots, \vec{t}_n} \psi \text{ iff there are } Y, Y' \subseteq X \text{ s.t. } Y \cup Y' = X, \mathcal{M} \models_Y \varphi, \mathcal{M} \models_{Y'} \psi \\ \text{and if } Y, Y' \neq \emptyset, \text{ then } Y(\vec{t}_i) = X(\vec{t}_i) = Y'(\vec{t}_i) \text{ for all } i \leq n. \end{aligned}$$

Before presenting a proof for this proposition, we explain its idea here briefly: We first check if the splitting can be done so that one of the sides is the empty team. In this case we don't set any requirements since all INEX_L -formulas are true in the empty team and on the other side values are trivially preserved since it has to be the whole team X .

Otherwise we fix two constants c_l, c_r which correspond to the left hand and right hand sides of the disjunction. Then we attach a “label” y to each assignment in the team. This label can be either c_l, c_r or both depending on if the assignment in question will be placed on the left, on the right or both. Since these labels are attached before doing the actual splitting, we can check beforehand that the information will be preserved.

The truth of formula θ_i guarantees that values of term t_i will be preserved on both sides for all values, except possibly for the value of \vec{c}_1 which is a constant. The formula θ'_i does the same, but it cannot make sure that the value for the constant \vec{c}_2 is preserved. But the truth of both θ_i and θ'_i guarantees that the values for \vec{t}_i are indeed preserved on both sides.

Proof. (Proposition 3.7) In this proof we use the abbreviation $\varphi \vee \psi := \varphi \vee_{\vec{t}_1, \dots, \vec{t}_n} \psi$. If X would be an empty team, the claim would hold trivially, and thus we may assume that $X \neq \emptyset$. By locality we may also assume that $c_l, c_r, y \notin \text{dom}(X)$. Suppose first that $\mathcal{M} \models_X \varphi \vee \psi$. Now either $\mathcal{M} \models_X \varphi \sqcup \psi$ or

$$\begin{aligned} \mathcal{M} \models_X \exists c_l \exists c_r (=(c_l) \wedge =(c_r) \wedge c_l \neq c_r \\ \wedge \exists y (((y = c_l \wedge \varphi) \vee (y = c_r \wedge \psi)) \wedge \bigwedge_{i \leq n} (\theta_i \wedge \theta'_i))). \quad (\star) \end{aligned}$$

Suppose first that $\mathcal{M} \models_X \varphi \sqcup \psi$, i.e. $\mathcal{M} \models_X \varphi$ or $\mathcal{M} \models_X \psi$. If $\mathcal{M} \models_X \varphi$, then we can choose $Y := X$ and $Y' := \emptyset$, when the claim holds trivially. Analogously

if $\mathcal{M} \models_X \psi$, we can choose $Y := \emptyset$ and $Y' := X$. Suppose then that (\star) holds. Now there exist $F_1 : X \rightarrow \mathcal{P}^*(M)$ and $F_2 : X[F_1/c_l] \rightarrow \mathcal{P}^*(M)$ such that

$$\begin{aligned} \mathcal{M} \models_{X_1} &= (c_l) \wedge = (c_r) \wedge c_l \neq c_r \\ &\wedge \exists y \left(((y = c_l \wedge \varphi) \vee (y = c_r \wedge \psi)) \wedge \bigwedge_{i \leq n} (\theta_i \wedge \theta'_i) \right), \end{aligned}$$

where $X_1 := X[F_1/c_l, F_2/c_r]$. Since $\mathcal{M} \models_{X_1} = (c_l)$, $\mathcal{M} \models_{X_1} = (c_r)$, $\mathcal{M} \models_{X_1} c_l \neq c_r$ and $X \neq \emptyset$, there exist $a, b \in M$ such that $X_1(c_l) = \{a\}$, $X_1(c_r) = \{b\}$ and $a \neq b$. There also exists a function $F_3 : X_1 \rightarrow \mathcal{P}^*(M)$ such that

$$\mathcal{M} \models_{X_2} ((y = c_l \wedge \varphi) \vee (y = c_r \wedge \psi)) \wedge \bigwedge_{i \leq n} (\theta_i \wedge \theta'_i), \text{ where } X_2 := X_1[F_3/y].$$

Now there exist $Z_1, Z'_1 \subseteq X_2$, such that $Z_1 \cup Z'_1 = X_2$, $\mathcal{M} \models_{Z_1} y = c_l \wedge \varphi$ and $\mathcal{M} \models_{Z'_1} y = c_r \wedge \psi$. Since $X_2(c_l) = \{a\}$, $X_2(c_r) = \{b\}$ and $a \neq b$, it is easy to see that the following holds for each $s \in X_2$:

$$s \in Z_1 \text{ iff } s(y) = a \quad \text{and} \quad s \in Z'_1 \text{ iff } s(y) = b.$$

Let $Y := Z_1 \upharpoonright \text{dom}(X)$ and $Y' := Z'_1 \upharpoonright \text{dom}(X)$. Since $\mathcal{M} \models_{Z_1} \varphi$ and $\mathcal{M} \models_{Z'_1} \psi$, we have $\mathcal{M} \models_Y \varphi$ and $\mathcal{M} \models_{Y'} \psi$ by locality. Because $Z_1 \cup Z'_1 = X_2$, we must also have $Y \cup Y' = X$ (recall that we assumed that $c_l, c_r, y \notin \text{dom}(X)$).

We still need to show that the values of \vec{t}_i ($i \leq n$) are preserved when X is split into Y and Y' . For the sake of showing this, let $i \leq n$, whence $\mathcal{M} \models_{X_2} \theta_i \wedge \theta'_i$. In particular $\mathcal{M} \models_{X_2} \theta_i$ and thus there are $\mathcal{F}_1 : X_2 \rightarrow \mathcal{P}^*(M^k)$ and $\mathcal{F}_2 : X_2[\mathcal{F}_1/\vec{z}_1] \rightarrow \mathcal{P}^*(M^k)$ such that

$$\begin{aligned} \mathcal{M} \models_{X_3} & \left((y = c_l \wedge \vec{z}_1 = \vec{t}_i \wedge \vec{z}_2 = \vec{c}_1) \right. \\ & \left. \vee (y = c_r \wedge \vec{z}_1 = \vec{c}_1 \wedge \vec{z}_2 = \vec{t}_i) \right) \wedge \vec{t}_i \subseteq \vec{z}_1 \wedge \vec{t}_i \subseteq \vec{z}_2, \end{aligned}$$

where $X_3 = X_2[\mathcal{F}_1/\vec{z}_1, \mathcal{F}_2/\vec{z}_2]$. Now there are $Z_2, Z'_2 \subseteq X_3$ s.t. $Z_2 \cup Z'_2 = X_3$ and

$$\begin{cases} \mathcal{M} \models_{Z_2} y = c_l \wedge \vec{z}_1 = \vec{t}_i \wedge \vec{z}_2 = \vec{c}_1 \\ \mathcal{M} \models_{Z'_2} y = c_r \wedge \vec{z}_1 = \vec{c}_1 \wedge \vec{z}_2 = \vec{t}_i. \end{cases}$$

Let $\vec{a} := (a, \dots, a)$ and $\vec{b} := (b, \dots, b)$. For the sake of showing that $X(\vec{t}_i) \subseteq Y(\vec{t}_i) \cup \{\vec{a}\}$, let $\vec{c} \in X(\vec{t}_i)$. Now there is $s \in X$ such that $s(\vec{t}_i) = \vec{c}$, whence there is $r \in X_3$ such that $r(\vec{t}_i) = s(\vec{t}_i)$. Since $\mathcal{M} \models_{X_3} \vec{t}_i \subseteq \vec{z}_1$, there exists $r' \in X_3$ such that $r'(\vec{z}_1) = r(\vec{t}_i)$. Now we have $\vec{c} = s(\vec{t}_i) = r(\vec{t}_i) = r'(\vec{z}_1)$.

Suppose first $r' \in Z_2$. Then $r'(\vec{z}_1) = r'(\vec{t}_i)$ and $r'(y) = r'(c_l) = a$. Hence there is $s' \in Y$ s.t. $s'(\vec{t}_i) = r'(\vec{t}_i)$. Now $\vec{c} = r'(\vec{z}_1) = r'(\vec{t}_i) = s'(\vec{t}_i) \in Y(\vec{t}_i)$. If $r' \notin Z_2$, then $r' \in Z'_2$, whence we have $\vec{c} = r'(\vec{z}_1) = r'(\vec{c}_1) = r'(c_l \dots c_l) = \vec{a}$. Hence in either case $\vec{c} \in Y(\vec{t}_i) \cup \{\vec{a}\}$ and thus $X(\vec{t}_i) \subseteq Y(\vec{t}_i) \cup \{\vec{a}\}$.

By using the fact that $\mathcal{M} \models_{X_2} \theta'_i$, we can analogously deduce the inclusion $X(\vec{t}_i) \subseteq Y(\vec{t}_i) \cup \{\vec{b}\}$. Since $\vec{a} \neq \vec{b}$, it thus has to be that $X(\vec{t}_i) \subseteq Y(\vec{t}_i)$. Clearly $Y(\vec{t}_i) \subseteq X(\vec{t}_i)$, and therefore we have $Y(\vec{t}_i) = X(\vec{t}_i)$. By using a symmetric argumentation we can also show that $Y'(\vec{t}_i) = X(\vec{t}_i)$.

Suppose then that there exist $Y, Y' \subseteq X$ such that $Y \cup Y' = X$, $\mathcal{M} \models_Y \varphi$ and $\mathcal{M} \models_{Y'} \psi$, and if $Y, Y' \neq \emptyset$, then we have $Y(\vec{t}_i) = Y'(\vec{t}_i) = X(\vec{t}_i)$ for each $i \leq n$.

If $Y = \emptyset$, then $Y' = X$ and thus $\mathcal{M} \models_X \psi$. Therefore $\mathcal{M} \models_X \varphi \sqcup \psi$ and thus $\mathcal{M} \models \varphi \vee \psi$. And if $Y' = \emptyset$, we obtain $\mathcal{M} \models \varphi \vee \psi$ by a similar argumentation. Hence we may assume $Y, Y' \neq \emptyset$, whence $Y(\vec{t}_i) = Y'(\vec{t}_i) = X(\vec{t}_i)$ for each $i \leq n$.

We first examine the special case when $|M| = 1$. Because $X \neq \emptyset$, the team X has to be a singleton $\{s\}$ for some s . Since $Y, Y' \neq \emptyset$, we have $Y = X$ and $Y' = X$. Therefore $\mathcal{M} \models_X \varphi \sqcup \psi$ and thus we have $\mathcal{M} \models_X \varphi \vee \psi$. Hence we may assume that $|M| \geq 2$, whence there are $a, b \in M$ such that $a \neq b$.

Let $F_1 : X \rightarrow \mathcal{P}^*(M)$ s.t. $s \mapsto \{a\}$ and let $F_2 : X[F_1/c_l] \rightarrow \mathcal{P}^*(M)$ s.t. $s \mapsto \{b\}$. We write $X_1 := X[F_1/c_l, F_2/c_r]$. By the definitions of F_1 and F_2 , we clearly have $\mathcal{M} \models_{X_1} = (c_l)$, $\mathcal{M} \models_{X_1} = (c_r)$ and $\mathcal{M} \models_{X_1} c_l \neq c_r$. Let

$$F_3 : X_1 \rightarrow \mathcal{P}^*(M) \text{ s.t. } \begin{cases} s \mapsto \{a\} & \text{if } s \upharpoonright \text{dom}(X) \in Y \setminus Y' \\ s \mapsto \{b\} & \text{if } s \upharpoonright \text{dom}(X) \in Y' \setminus Y \\ s \mapsto \{a, b\} & \text{if } s \upharpoonright \text{dom}(X) \in Y \cap Y'. \end{cases}$$

We define the following teams $X_2 := X_1[F_3/y]$, $Z_1 := \{s \in X_3 \mid s(y) = a\}$ and $Z'_1 := \{s \in X_3 \mid s(y) = b\}$. Now it clearly holds that $Z_1 \cup Z'_1 = X_2$, $\mathcal{M} \models_{Z_1} y = c_l$ and $\mathcal{M} \models_{Z'_1} y = c_r$. By locality and the definition of F_3 , we have $\mathcal{M} \models_{Z_1} \varphi$ and $\mathcal{M} \models_{Z'_1} \psi$. Therefore $\mathcal{M} \models_{X_2} (y = c_l \wedge \varphi) \vee (y = c_r \wedge \psi)$.

Let $i \leq n$. We define $\vec{a} := (a, \dots, a)$ and

$$\begin{cases} \mathcal{F}_1 : X_2 \rightarrow \mathcal{P}^*(M^k) \text{ s.t. } \begin{cases} s \mapsto \{s(\vec{t}_i)\} & \text{if } s(y) = a \\ s \mapsto \{\vec{a}\} & \text{if } s(y) = b \end{cases} \\ \mathcal{F}_2 : X_2[\mathcal{F}_1/\vec{z}_1] \rightarrow \mathcal{P}^*(M^k) \text{ s.t. } \begin{cases} s \mapsto \{\vec{a}\} & \text{if } s(y) = a \\ s \mapsto \{s(\vec{t}_i)\} & \text{if } s(y) = b. \end{cases} \end{cases}$$

Let $X_3 := X_2[\mathcal{F}_1/\vec{z}_1, \mathcal{F}_2/\vec{z}_2]$, $Z_2 := \{s \in X_3 \mid s(y) = a\}$ and $Z'_2 := \{s \in X_3 \mid s(y) = b\}$. Now $Z_2 \cup Z'_2 = X_3$ and by the definitions of \mathcal{F}_1 and \mathcal{F}_2 we have

$$\mathcal{M} \models_{Z_2} y = c_l \wedge \vec{z}_1 = \vec{t}_i \wedge \vec{z}_2 = \vec{c}_1 \quad \text{and} \quad \mathcal{M} \models_{Z'_2} y = c_r \wedge \vec{z}_1 = \vec{c}_1 \wedge \vec{z}_2 = \vec{t}_i.$$

For the sake of showing that $\mathcal{M} \models_{X_3} \vec{t}_i \subseteq \vec{z}_1$, let $r \in X_3$. Now there is $s \in X$, s.t. $r(\vec{t}_i) = s(\vec{t}_i)$. Since $s(\vec{t}_i) \in X(\vec{t}_i) = Y(\vec{t}_i)$, there is $s' \in Y$, such that $s'(\vec{t}_i) = s(\vec{t}_i)$. Let $r' := s'[a/c_l, b/c_r, a/y, s'(\vec{t}_i)/\vec{z}_1, \vec{a}/\vec{z}_2]$. Now $r' \in X_3$ and $r(\vec{t}_i) = s(\vec{t}_i) = s'(\vec{t}_i) = r'(\vec{z}_1)$. Hence $\mathcal{M} \models_{X_3} \vec{t}_i \subseteq \vec{z}_1$. Analogously we can show

that $\mathcal{M} \models_{X_3} \vec{t}_i \subseteq \vec{z}_2$ and thus $\mathcal{M} \models_{X_2} \theta_i$. By a similar argumentation $\mathcal{M} \models_{X_2} \theta'_i$ and thus $\mathcal{M} \models_{X_2} \bigwedge_{i \leq n} (\theta_i \wedge \theta'_i)$. Hence (\star) holds, and therefore $\mathcal{M} \models_X \varphi \vee \psi$. \square

Term value preserving disjunction has several natural variants. The version we defined requires that the values of given tuples of terms are preserved both on the left and right side of the disjunction. We could weaken this condition by requiring these values to be preserved only on the left, only on the right or only on either of the sides without specifying which. Or we could modify this condition by requiring different tuples of terms to be preserved on the left and different tuples to be preserved on the right.

Now we allow the splitting to be done in such a way that either of the sides becomes empty, which is natural for our needs since INEX has empty team property. But strictly speaking, the values of the given terms are not necessarily preserved in this case, since there are no values in the empty team. If we require values to be preserved in then as well, we can additionally require that splitting must be done in a way that neither of the sides becomes empty. If we only require this condition – ignoring the values of any terms – we obtain a disjunction that can be seen as a dual operator for intuitionistic disjunction⁴.

We will not go into details here, but all of the variants described above can be defined in INEX. We just need to do some simple modifications on the formula that defines term value preserving disjunction in Definition 3.7. In this paper we use term value preserving disjunction only as a useful tool in INEX, but it would be interesting to study the properties and the expressive power of this operator (or some of its variants) independently.

3.5. Relativization method for team semantics

In this subsection we introduce an application which uses several of the new operators that we have defined in this section. Suppose that φ is an INEX_L -sentence and $y \notin \text{Vr}(\varphi)$ is a variable in the domain of a team X . If we replace all quantifiers $\exists x, \forall x$ in φ with the corresponding inclusion quantifiers $(\exists x \subseteq y)$, $(\forall x \subseteq y)$, the evaluation of the resulting formula is identical to evaluation of φ , except that the quantifiers in φ may only choose values within the values of y . If we further replace disjunctions in φ with the ones that preserve the value of y , then the quantifications may only choose values within the set $X(y)$ (the *initial* values of y in X). Since the resulting formula only “sees” the part of model that is restricted to the set $X(y)$, we call this method *relativization*.

Definition 3.8. Let φ be an INEX_L -sentence and let $y \notin \text{Vr}(\varphi)$ be a variable.

⁴Intuitionistic disjunction states that the splitting must be done in a way that either of the sides becomes empty – a dual condition is that neither of the sides can be left empty.

The relativization of φ on y , denoted by $\varphi \upharpoonright y$, is defined recursively:

$$\begin{aligned}
\psi \upharpoonright y &= \psi \quad \text{if } \psi \text{ is a literal or inclusion/exclusion atom} \\
(\psi \wedge \theta) \upharpoonright y &= \psi \upharpoonright y \wedge \theta \upharpoonright y \\
(\psi \vee \theta) \upharpoonright y &= \psi \upharpoonright y \vee \theta \upharpoonright y, \quad \text{where } \vee := \bigvee_y \\
(\exists x \psi) \upharpoonright y &= (\exists x \subseteq y)(\psi \upharpoonright y) \\
(\forall x \psi) \upharpoonright y &= (\forall x \subseteq y)(\psi \upharpoonright y).
\end{aligned}$$

Note that since φ is a sentence, we have $\text{Fr}(\varphi \upharpoonright y) = \{y\}$.

Let X be a team and $y \in \text{dom}(X) \setminus \text{Vr}(\varphi)$. If φ defines some property of the domain of a model, then the formula $\varphi \upharpoonright y$ defines the same property of the set values for y in the team X . This is proven in Proposition 3.8 below. This proposition could be proven also for many other logics \mathcal{L} with team semantics. If the following assumptions hold for \mathcal{L} , the proof can be done identically as it is done here: \mathcal{L} is an extension of FO with new atomic formulas, it is local, has empty team property, and inclusion quantifiers (for single variables) and term value preserving disjunction (for single terms) can be expressed in \mathcal{L} . Note that in order to express these operators, it would suffice that we could use *unary* inclusion and exclusion atoms in \mathcal{L} .

If $\mathcal{M} = (\mathcal{I}, M)$ is an L -model and $A \subseteq M$, the notation $\mathcal{M} \upharpoonright A$ denotes the submodel of \mathcal{M} that is *relativized* on A . That is, the universe of $\mathcal{M} \upharpoonright A$ is the set A and the symbols in L are interpreted as: $R^{\mathcal{M} \upharpoonright A} = R^{\mathcal{M}} \upharpoonright A^n$ for n -ary relation symbols $R \in L$, $f^{\mathcal{M} \upharpoonright A} = f^{\mathcal{M}} \upharpoonright A^n$ for n -ary function symbols $f \in L$ and $c^{\mathcal{M} \upharpoonright A} = c^{\mathcal{M}}$ for constant symbols $c \in L$. Note that if L contains function or constant symbols, then $\mathcal{M} \upharpoonright A$ can be an L -model only if $f^{\mathcal{M}} \upharpoonright A^n : A^n \rightarrow A$ for all each n -ary $f \in L$ and $c^{\mathcal{M}} \in A$ for each $c \in L$. But if L is relational, then $\mathcal{M} \upharpoonright A$ is an L -model for any $A \subseteq M$.

Proposition 3.8. *Let φ be an INEX_L -sentence and y be a variable such that $y \notin \text{Vr}(\varphi)$. Now*

$$\mathcal{M} \models_X \varphi \upharpoonright y \quad \text{iff} \quad \mathcal{M} \upharpoonright X(y) \models \varphi$$

for all L -models \mathcal{M} and teams X such that $\mathcal{M} \upharpoonright X(y)$ is an L -model.

Proof. We first show that

$$\text{If } \mathcal{M} \models_X \mu \upharpoonright y, \text{ then } \mathcal{M} \upharpoonright X(y) \models_X \mu, \tag{R1}$$

for all $\mu \in \text{Sf}(\varphi)$ and teams X for which the following condition holds:

$$X(z) \subseteq X(y) \text{ for all } z \in \text{dom}(X). \tag{\star}$$

Note that if the condition (\star) would not hold, then X would not be a team for the model $\mathcal{M} \upharpoonright X(y)$. We prove the claim (R1) by induction on μ :

- If μ is a literal or inclusion/exclusion atom, then the claim holds trivially since $\mu \upharpoonright y = \mu$ and $X(z) \subseteq X(y)$ for all $z \in \text{Vr}(\mu)$.
- The case $\mu = \psi \wedge \theta$ is straightforward to prove.
- Let $\mu = \psi \vee \theta$. Suppose first that $\mathcal{M} \models_X (\psi \vee \theta) \upharpoonright y$, i.e. $\mathcal{M} \models_X \psi \upharpoonright y \vee \theta \upharpoonright y$. Thus there exist $Y_1, Y_2 \subseteq X$ s.t. $Y_1 \cup Y_2 = X$, $\mathcal{M} \models_{Y_1} \psi \upharpoonright y$ and $\mathcal{M} \models_{Y_2} \theta \upharpoonright y$, and if $Y_1, Y_2 \neq \emptyset$, then $Y_1(y) = Y_2(y) = X(y)$. If $Y_1 = \emptyset$, then the condition (\star) holds trivially for Y_1 . Also if $Y_2 = \emptyset$, then (\star) holds for Y_1 as $Y_1 = X$. Suppose then that $Y_1, Y_2 \neq \emptyset$, whence $Y_1(y) = X(y)$. Since $Y_1 \subseteq X$, we have $Y_1(z) \subseteq X(z) \subseteq X(y) = Y_1(y)$ for all $z \in \text{dom}(X) = \text{dom}(Y_1)$. Thus (\star) holds for Y_1 in all cases. By an analogous argumentation (\star) holds for Y_2 .

Suppose first that $Y_2 = \emptyset$. Now $Y_1 = X$ and thus $\mathcal{M} \models_X \psi \upharpoonright y$. By the inductive hypothesis $\mathcal{M} \upharpoonright X(y) \models_X \psi$ and thus $\mathcal{M} \upharpoonright X(y) \models_X \psi \vee \theta$. The case when $Y_1 = \emptyset$ is analogous. Suppose then that $Y_1, Y_2 \neq \emptyset$, whence $Y_1(y) = Y_2(y) = X(y)$. By the inductive hypothesis $\mathcal{M} \upharpoonright Y_1(y) \models_{Y_1} \psi$. Since $Y_1(y) = X(y)$, we have $\mathcal{M} \upharpoonright X(y) \models_{Y_1} \psi$. By similar argumentation we have $\mathcal{M} \upharpoonright X(y) \models_{Y_2} \theta$ and thus $\mathcal{M} \upharpoonright X(y) \models_X \psi \vee \theta$.

- Let $\mu = \exists x \psi$. Suppose $\mathcal{M} \models_X (\exists x \psi) \upharpoonright y$, i.e. $\mathcal{M} \models_X (\exists x \subseteq y)(\psi \upharpoonright y)$. Thus there exists $F : X \rightarrow \mathcal{P}^*(X(y))$ such that $\mathcal{M} \models_{X'} \psi \upharpoonright y$, where $X' = X[F/x]$. Since $X(z) \subseteq X(y) = X'(y)$ for all $z \in \text{dom}(X)$ and $X'(x) \subseteq X(y) = X'(y)$, the condition (\star) holds for the team X' . Thus, by the inductive hypothesis, $\mathcal{M} \upharpoonright X'(y) \models_{X'} \psi$. Since $X'(y) = X(y)$, we have $\mathcal{M} \upharpoonright X(y) \models_{X[F/x]} \psi$ and furthermore $\mathcal{M} \upharpoonright X(y) \models_X \exists x \psi$.
- The case $\mu = \forall x \psi$ can be proven similarly as the previous case (see [17]).

We then show that if $A \subseteq M$ such that $\mathcal{M} \upharpoonright A$ is an L -model, then the following holds:

$$\text{If } \mathcal{M} \upharpoonright A \models_X \mu, \text{ then } \mathcal{M} \models_{X[A/y]} \mu \upharpoonright y, \quad (\text{R2})$$

for all $\mu \in \text{Sf}(\varphi)$ and teams X for which $\text{dom}(X) = \text{Fr}(\mu)$. We prove this claim by induction on μ :

- Suppose that μ is a literal or inclusion/exclusion atom. Since X is a team for the model $\mathcal{M} \upharpoonright A$, we must have $X(z) \subseteq A$ for all $z \in \text{dom}(X) = \text{Vr}(\mu)$. We also have $\mu \upharpoonright y = \mu$ and $y \notin \text{Fr}(\mu)$, and thus the claim holds trivially.
- The case $\mu = \psi \wedge \theta$ is straightforward to prove.
- Let $\mu = \psi \vee \theta$. Suppose that $\mathcal{M} \upharpoonright A \models_X \psi \vee \theta$, i.e. there are $Y_1, Y_2 \subseteq X$ such that $Y_1 \cup Y_2 = X$, $\mathcal{M} \upharpoonright A \models_{Y_1} \psi$ and $\mathcal{M} \upharpoonright A \models_{Y_2} \theta$. Hence, by the inductive hypothesis and locality, $\mathcal{M} \models_{Y'_1} \psi \upharpoonright y$ and $\mathcal{M} \models_{Y'_2} \theta \upharpoonright y$, where $Y'_1 = Y_1[A/y]$

and $Y'_2 = Y_2[A/y]$. Now clearly $Y'_1 \cup Y'_2 = X[A/y]$ and if $Y'_1, Y'_2 \neq \emptyset$, then

$$Y'_1(y) = Y'_2(y) = A = (X[A/y])(y).$$

Thus we have $\mathcal{M} \models_{X[A/y]} \psi \upharpoonright y \sqcup \theta \upharpoonright y$, i.e. $\mathcal{M} \models_{X[A/y]} (\psi \vee \theta) \upharpoonright y$.

- Let $\mu = \exists x \psi$. Suppose $\mathcal{M} \upharpoonright A \models_X \exists x \psi$, i.e. there is $F : X \rightarrow \mathcal{P}^*(A)$ such that $\mathcal{M} \upharpoonright A \models_{X'} \psi$, where $X' = X[F/x]$. Thus, by the inductive hypothesis and locality, $\mathcal{M} \models_{X'[A/y]} \psi \upharpoonright y$. Let

$$F' : X[A/y] \rightarrow \mathcal{P}^*(A), \quad s \mapsto F(s \upharpoonright \text{Fr}(\mu)) \quad \text{and} \quad X'' := (X[A/y])[F'/x].$$

Note that F' is well-defined since $\text{dom}(X) = \text{Fr}(\mu)$ by the assumption. By the definition of F' , we have $X'' = X'[A/y]$ and thus $\mathcal{M} \models_{X''} \psi \upharpoonright y$. We also have $\text{ran}(F') = \text{ran}(F) \subseteq \mathcal{P}^*(A) = \mathcal{P}^*((X[A/y])(y))$, and therefore $\mathcal{M} \models_{X[A/y]} (\exists x \subseteq y)(\psi \upharpoonright y)$, i.e. $\mathcal{M} \models_{X[A/y]} (\exists x \psi) \upharpoonright y$.

- The case $\mu = \forall x \psi$ can be proven similarly as the previous case (see [17]).

We are now ready to prove the claim of this proposition:

$$\mathcal{M} \models_X \varphi \upharpoonright y \quad \text{iff} \quad \mathcal{M} \upharpoonright X(y) \models \varphi.$$

Suppose first that $\mathcal{M} \models_X \varphi \upharpoonright y$. By locality we have $\mathcal{M} \models_{X'} \varphi \upharpoonright y$, where $X' = X \upharpoonright \text{Fr}(\varphi \upharpoonright y)$. Since φ is a sentence, $\text{dom}(X') = \text{Fr}(\varphi \upharpoonright y) = \{y\}$, and thus the condition (\star) holds trivially for the team X' . Hence by (R1) we have $\mathcal{M} \upharpoonright X'(y) \models_{X'} \varphi$. Since $X'(y) = X(y)$, we have $\mathcal{M} \upharpoonright X(y) \models_{X'} \varphi$ and thus by locality $\mathcal{M} \upharpoonright X(y) \models \varphi$.

Suppose then that $\mathcal{M} \upharpoonright X(y) \models \varphi$. Now by (R2), we have $\mathcal{M} \models_{\{\emptyset\}[X(y)/y]} \varphi \upharpoonright y$. Since $X \upharpoonright \{y\} = \{\emptyset\}[X(y)/y]$, we have $\mathcal{M} \models_{X \upharpoonright \{y\}} \varphi \upharpoonright y$. Since $\text{Fr}(\varphi \upharpoonright y) = \{y\}$, by locality $\mathcal{M} \models_X \varphi \upharpoonright y$. \square

The relativization method gives us a simple way to express properties of certain sets of values in a team. We can apply the same technique for many other logics with team semantics if we extend them with unary inclusion and exclusion atoms. For example, there is a dependence logic sentence φ which expresses that a model has even cardinality ([18]). Now the formula $\varphi \upharpoonright y$ expresses that the variable y has even number of different values in a team. We will give more examples on this method in Section 5.

4. The expressive power of k -ary inclusion-exclusion logic

In this section we will analyze the expressive power of $\text{INEX}[k]$. We first present translations from $\text{INC}[k]$ and $\text{EXC}[k]$ to $\text{ESO}[k]$ and then combine them to form a translation from $\text{INEX}[k]$ to $\text{ESO}[k]$. For the other direction we show

that any $\text{ESO}[k]$ -formula, with at most k -ary free relation variables, can be expressed in $\text{INEX}[k]$.

4.1. Translation from $\text{INEX}[k]$ to $\text{ESO}[k]$

For the language ESO_L we also need a set of *relation variables* which are symbols not in the vocabulary L . These relation variables can appear in atomic formulas similarly as relation symbols in L and they can also be existentially quantified. We require all of these *second order quantifiers* to appear in front of the ESO_L -formula, before its *first order part*.

In the language $\text{ESO}_L[k]$ we only allow existential quantification of at most k -ary relation variables, but free relation variables in a formula may have any arity. Hence $\text{ESO}_L[0]$ -formulas are second order quantifier free, but may contain free relation variables. If an ESO_L -formula Φ has free relation variables R_1, \dots, R_n , we can emphasize this by writing Φ as $\Phi(R_1 \dots R_n)$. In this paper we will not consider ESO -formulas with free first order variables and thus their first order part can be seen as FO-sentence. After evaluating all second order quantifications, the truth of Φ in depends only on the first order part of Φ . We may then apply team semantics for the first order part of Φ in any suitable model, whence flatness and locality properties hold as well.

Let \mathcal{L} be any logic with team semantics and let $\varphi(\vec{y})$ be an \mathcal{L} -formula. The truth of φ depends on a model \mathcal{M} and a team X . If \mathcal{L} is local, it is sufficient to consider the team $X \upharpoonright \text{Vr}(\vec{y})$ that is determined by the relation $X(\vec{y})$ (notice that the ordering of the variables in \vec{y} here needs to be *fixed*). Therefore it is natural to compare φ with an ESO -formula $\Phi(R)$ and check whether the relations in \mathcal{M} that satisfy Φ correspond to the relations $X(\vec{y})$, where X satisfies φ . Thus we say that φ and Φ are equivalent if we have

$$\mathcal{M} \models_X \varphi \text{ iff } \mathcal{M}[X(\vec{y})/R] \models \Phi.$$

Translation from $\text{EXC}[k]$ to $\text{ESO}[k]$

In the next theorem we formulate a translation from $\text{EXC}[k]$ to $\text{ESO}[k]$. The idea of the proof is that we quantify a separate relation variable P for each occurrence of an exclusion atom $\vec{t}_1 \mid \vec{t}_2$. The values quantified for P are the limit for the values that \vec{t}_1 can get and \vec{t}_2 cannot get, when $\vec{t}_1 \mid \vec{t}_2$ is evaluated.

Theorem 4.1. *Let $\varphi(\vec{y}) \in \text{EXC}_L[k]$. Now there exists an $\text{ESO}_L[k]$ -formula $\Phi(R)$, for which*

$$\mathcal{M} \models_X \varphi \text{ iff } \mathcal{M}[X(\vec{y})/R] \models \Phi.$$

Proof. Without loss of generality we may assume that each exclusion atom in φ is k -ary. We index these atoms by $(\vec{t}_1 \mid \vec{t}_2)_1, \dots, (\vec{t}_1 \mid \vec{t}_2)_n$. This is done so that each occurrence of an exclusion atom has a unique index. Let P_1, \dots, P_n be

k -ary relation variables. Let $\psi \in \text{Sf}(\varphi)$. We define the formula ψ' recursively:

$$\begin{aligned}\psi' &= \psi \text{ if } \psi \text{ is a literal} \\ ((\vec{t}_1 \mid \vec{t}_2)_i)' &= P_i \vec{t}_1 \wedge \neg P_i \vec{t}_2 \text{ for each } i \leq n \\ (\psi \wedge \theta)' &= \psi' \wedge \theta', \quad (\psi \vee \theta)' = \psi' \vee \theta' \\ (\exists x \psi)' &= \exists x \psi', \quad (\forall x \psi)' = \forall x \psi'.\end{aligned}$$

We can now define the formula Φ in the following way⁵:

$$\Phi := \exists P_1 \dots \exists P_n \forall \vec{y} (\neg R\vec{y} \vee (R\vec{y} \wedge \varphi')).$$

Clearly Φ is an $\text{ESO}_L[k]$ -formula and R is the only free relation variable in Φ . We first need to prove the following claim.

Claim 2. *Let $\mu \in \text{Sf}(\varphi)$. Now the following holds for all suitable teams X :*

$$\mathcal{M} \models_X \mu \text{ iff there exist } A_1, \dots, A_n \subseteq M^k \text{ such that } \mathcal{M}' \models_X \mu',$$

where $\mathcal{M}' := \mathcal{M}[\vec{A}/\vec{P}]$ ($= \mathcal{M}[A_1/P_1, \dots, A_n/P_n]$).

We prove this claim by structural induction on μ :

- If μ is a literal we can set $A_i := \emptyset$ for each $i \leq n$. Now the claim holds trivially since $\mu' = \mu$ and P_i does not occur in μ for any $i \leq n$.
- Let $\varphi = (\vec{t}_1 \mid \vec{t}_2)_j$ for some $j \leq n$. Suppose first that $\mathcal{M} \models_X \vec{t}_1 \mid \vec{t}_2$. Let

$$\mathcal{M}' := \mathcal{M}[\vec{A}/\vec{P}], \text{ where } A_i := \begin{cases} X(\vec{t}_1) & \text{if } i = j \\ \emptyset & \text{else.} \end{cases} \quad (i \leq n)$$

Because $X(\vec{t}_1) = A_j = P_j^{\mathcal{M}'}$, we clearly have $\mathcal{M}' \models_X P_j \vec{t}_1$.

For the sake of contradiction, suppose that there exists $s \in X$ for which $s(\vec{t}_2) \in P_j^{\mathcal{M}'}$. Since $P_j^{\mathcal{M}'} = X(\vec{t}_1)$, there exists $s' \in X$ s.t. $s'(\vec{t}_1) = s(\vec{t}_2)$. But this is a contradiction since by the assumption $\mathcal{M} \models_X \vec{t}_1 \mid \vec{t}_2$. Therefore $\mathcal{M}' \models_X \neg P_j \vec{t}_2$ and thus $\mathcal{M}' \models_X P_j \vec{t}_1 \wedge \neg P_j \vec{t}_2$, i.e. $\mathcal{M}' \models_X ((\vec{t}_1 \mid \vec{t}_2)_j)'$.

Suppose then that there exist $A_1, \dots, A_n \subseteq M^k$ s.t. $\mathcal{M}' \models_X ((\vec{t}_1 \mid \vec{t}_2)_j)'$. Hence $\mathcal{M}' \models_X P_j \vec{t}_1$ and $\mathcal{M}' \models_X \neg P_j \vec{t}_2$. For the sake of contradiction, suppose that there are $s, s' \in X$ s.t. $s(\vec{t}_1) = s'(\vec{t}_2)$. Because $\mathcal{M}' \models_X P_j \vec{t}_1$, we have $s(\vec{t}_1) \in P_j^{\mathcal{M}'}$. But because $\mathcal{M}' \models_X \neg P_j \vec{t}_2$, it has to be that $s(\vec{t}_1) = s'(\vec{t}_2) \notin P_j^{\mathcal{M}'}$. This is a contradiction, and thus $\mathcal{M} \models_X \vec{t}_1 \mid \vec{t}_2$.

⁵If φ is an EXC_L -sentence we define simply $\Phi := \exists P_1 \dots \exists P_n \varphi'$.

- Let $\mu = \psi \vee \theta$ (The case $\mu = \psi \wedge \theta$ can be proven similarly). Suppose first that $\mathcal{M} \models_X \psi \vee \theta$. Thus there are $Y, Y' \subseteq X$ s.t. $Y \cup Y' = X$, $\mathcal{M} \models_Y \psi$ and $\mathcal{M} \models_{Y'} \theta$. By the inductive hypothesis there are $B_1, \dots, B_n \subseteq M^k$ and $B'_1, \dots, B'_n \subseteq M^k$ s.t. $\mathcal{M}[\vec{B}/\vec{P}] \models_Y \psi'$ and $\mathcal{M}[\vec{B}'/\vec{P}] \models_{Y'} \theta'$. Let

$$\mathcal{M}' := \mathcal{M}[\vec{A}/\vec{P}], \text{ where } A_i := \begin{cases} B_i & \text{if } P_i \text{ occurs in } \psi' \\ B'_i & \text{if } P_i \text{ does not occur in } \psi'. \end{cases}$$

Since none of P_i can occur in both ψ' and θ' , we have $\mathcal{M}' \models_Y \psi'$ and $\mathcal{M}' \models_{Y'} \theta'$. Hence $\mathcal{M}' \models_X \psi' \vee \theta'$, i.e. $\mathcal{M}' \models_X (\psi \vee \theta)'$.

Suppose then that there are $A_1, \dots, A_n \subseteq M^k$ s.t. $\mathcal{M}' \models_X (\psi \vee \theta)'$. Thus $\mathcal{M}' \models_X \psi' \vee \theta'$, i.e. there are $Y, Y' \subseteq X$ s.t. $Y \cup Y' = X$, $\mathcal{M}' \models_Y \psi'$ and $\mathcal{M}' \models_{Y'} \theta'$. By the inductive hypothesis $\mathcal{M} \models_Y \psi$ and $\mathcal{M} \models_{Y'} \theta$, i.e. $\mathcal{M} \models_X \psi \vee \theta$.

- The cases $\mu = \exists x \psi$ and $\mu = \forall x \psi$ are straightforward to prove.

Let $\mathcal{M}' = \mathcal{M}[\vec{A}/\vec{P}]$ for some $A_1, \dots, A_n \subseteq M^k$. Since $\text{Fr}(\varphi') = \text{Vr}(\vec{y})$, by locality it is easy to see that the following holds for all suitable teams X :

$$\mathcal{M}' \models_X \varphi' \text{ iff } \mathcal{M}'[X(\vec{y})/R] \models \forall \vec{y} (\neg R\vec{y} \vee (R\vec{y} \wedge \varphi')).$$

By combining this with the result of Claim 2, we obtain:

$$\begin{aligned} \mathcal{M} \models_X \varphi \text{ iff there are } A_1, \dots, A_n \subseteq M^k \\ \text{s.t. } \mathcal{M}[\vec{A}/\vec{P}, X(\vec{y})/R] \models \forall \vec{y} (\neg R\vec{y} \vee (R\vec{y} \wedge \varphi')). \end{aligned}$$

Equivalently: $\mathcal{M} \models_X \varphi$ iff $\mathcal{M}[X(\vec{y})/R] \models \Phi$. □

Translation from INC[k] to ESO[k]

In the next theorem we present a translation from INC[k] to ESO[k]. Again the idea is that we quantify a separate predicate symbol P for each inclusion atom $\vec{t}_1 \subseteq \vec{t}_2$, and the values of \vec{t}_1 must be included in the values chosen for P . But we must also show that each value of P is a value that tuple \vec{t}_2 gets in the team when $\vec{t}_1 \subseteq \vec{t}_2$ is evaluated. For this we need special formulas, $\varphi'_i(\vec{u})$, which “find” the assignment that gets same values for \vec{u} and \vec{t}_2 – for any value of \vec{u} that is in the values chosen for P .

Theorem 4.2. *Let $\varphi(\vec{y})$, where $\vec{y} = y_1 \dots y_m$, be an INC_L[k]-formula. Then there exists an ESO_L[k]-formula $\Phi(R)$, for which we have*

$$\mathcal{M} \models_X \varphi \text{ iff } \mathcal{M}[X(\vec{y})/R] \models \Phi.$$

Proof. Without loss of generality we may assume that each inclusion atom in φ is k -ary. We index these atoms by $(\vec{t}_1 \subseteq \vec{t}_2)_1, \dots, (\vec{t}_1 \subseteq \vec{t}_2)_n$. Let \vec{u} be a k -tuple

of fresh variables and P_1, \dots, P_n be k -ary relation variables.

Let $\psi \in \text{Sf}(\varphi)$. We define the formula ψ' recursively:

$$\begin{aligned} (\psi)' &= \psi \text{ if } \psi \text{ is a literal} \\ ((\vec{t}_1 \subseteq \vec{t}_2)_i)' &= P_i \vec{t}_1 \text{ for each } i \leq n \\ (\psi \wedge \theta)' &= \psi' \wedge \theta', \quad (\psi \vee \theta)' = \psi' \vee \theta' \\ (\exists x \psi)' &= \exists x \psi', \quad (\forall x \psi)' = \forall x \psi'. \end{aligned}$$

Formulas ψ'_i are defined recursively for all $i \leq n$:

$$\begin{aligned} (\psi)'_i &= \psi \text{ if } \psi \text{ is a literal} \\ ((\vec{t}_1 \subseteq \vec{t}_2)_j)'_i &= P_j \vec{t}_1 \text{ if } j \neq i \\ ((\vec{t}_1 \subseteq \vec{t}_2)_i)'_i &= (\vec{u} = \vec{t}_2) \wedge P_i \vec{t}_1 \\ (\psi \wedge \theta)'_i &= \psi'_i \wedge \theta'_i \\ (\psi \vee \theta)'_i &= \begin{cases} \psi'_i & \text{if } (\vec{t}_1 \subseteq \vec{t}_2)_i \text{ occurs in } \psi \\ \theta'_i & \text{if } (\vec{t}_1 \subseteq \vec{t}_2)_i \text{ occurs in } \theta \\ \psi'_i \vee \theta'_i & \text{else} \end{cases} \\ (\exists x \psi)'_i &= \exists x \psi'_i \\ (\forall x \psi)'_i &= \exists x \psi'_i \wedge \forall x \psi'. \end{aligned}$$

Note that the cases of disjunction above are exclusive, since for each $i \leq n$ the inclusion atom $(\vec{t}_1 \subseteq \vec{t}_2)_i$ can occur in at most one of the disjuncts.

We can now define the formula Φ in the following way⁶:

$$\Phi := \exists P_1 \dots \exists P_n \left(\forall \vec{y} \left(\neg R \vec{y} \vee (R \vec{y} \wedge \varphi') \right) \wedge \bigwedge_{i \leq n} \forall \vec{u} \left(\neg P_i \vec{u} \vee \exists \vec{y} (R \vec{y} \wedge \varphi'_i(\vec{u})) \right) \right).$$

Clearly Φ is an $\text{ESO}_L[k]$ -formula and R is the only free relation variable in Φ . To complete the proof, we need to prove the following claim which demonstrates the relevance of the formulas φ'_i :

Claim 3. *The following holds for all $\mu \in \text{Sf}(\varphi)$ and all suitable teams X :*

$$\begin{aligned} \mathcal{M} \models_X \mu \quad &\text{iff} \quad \text{there exist } A_1, \dots, A_n \subseteq M^k \text{ s.t. } \mathcal{M}[\vec{A}/\vec{P}] \models_X \mu', \\ &\text{and for any } i \leq n \text{ and tuple of elements } \vec{a} \in A_i \\ &\text{there exists } s \in X \text{ s.t. } \mathcal{M}[\vec{A}/\vec{P}] \models_{\{s[\vec{a}/\vec{u}]\}} \mu'_i. \end{aligned}$$

We present a proof for this claim in the appendix.

⁶If φ is an INC_L -sentence we can define $\Phi := \exists P_1 \dots \exists P_n (\varphi' \wedge \bigwedge_{i \leq n} \forall \vec{u} (\neg P_i \vec{u} \vee \varphi'_i(\vec{u})))$.

Note that since μ' and μ'_i do not contain the relation variable R , we can replace the model $\mathcal{M}[\vec{A}/\vec{P}]$ with the model $\mathcal{M}[\vec{A}/\vec{P}, X(\vec{y})/R]$ in Claim 3. The existence of $s \in X$ for each tuple $\vec{a} \in A_i$ such that $s[\vec{a}/\vec{u}]$ satisfies φ'_i , guarantees that the values in A_i are included in the values \vec{t}_2 in the team when the inclusion atom $(\vec{t}_1 \subseteq \vec{t}_2)_i$ is evaluated. With this result we can prove the claim of this theorem; that is $\mathcal{M} \models_X \varphi$ iff $\mathcal{M}[X(\vec{y})/R] \models \Phi$.

Suppose first that $\mathcal{M} \models_X \varphi$. Now by Claim 3 there are $A_1, \dots, A_n \subseteq M^k$ s.t. $\mathcal{M}' \models_X \varphi'$, and for all $i \leq n$ and $\vec{a} \in A_i$ there exists $s \in X$ such that $\mathcal{M}' \models_{\{s[\vec{a}/\vec{u}]\}} \varphi'_i$, where $\mathcal{M}' = \mathcal{M}[\vec{A}/\vec{P}, X(\vec{y})/R]$. Let $j \leq n$ and let

$$Y := \{r \in \{\emptyset\}[M^k/\vec{u}] \mid r(\vec{u}) \notin A_j\} \text{ and } Y' := \{r \in \{\emptyset\}[M^k/\vec{u}] \mid r(\vec{u}) \in A_j\}.$$

Now $Y \cup Y' = \{\emptyset\}[M^k/\vec{u}]$ and because $A_j = P_j^{\mathcal{M}'}$, clearly $\mathcal{M}' \models_Y \neg P_j \vec{u}$. By the definition of Y' , we have $r(\vec{u}) \in A_j$ for each $r \in Y'$. Hence, by applying the result of Claim 3 for the values $r(\vec{u})$, the following holds: for each $r \in Y'$ there exists $s_r \in X$ s.t. $\mathcal{M}' \models_{\{s_r[r(\vec{u})/\vec{u}]\}} \varphi'_j$.

Let $\mathcal{F} : Y' \rightarrow \mathcal{P}^*(M^m)$ s.t. $r \mapsto \{s_r(\vec{y})\}$. Now $r[s_r(\vec{y})/\vec{y}] = s_r[r(\vec{u})/\vec{u}]$ for each $r \in Y'$ and thus $\mathcal{M}' \models_{\{r[s_r(\vec{y})/\vec{y}]\}} \varphi'_j$ for each $r \in Y'$. Hence by locality and flatness $\mathcal{M}' \models_{Y'[\mathcal{F}/\vec{y}]} \varphi'_j$. Because $s_r(\vec{y}) \in X(\vec{y}) = R^{\mathcal{M}'}$ for each $r \in Y'$, by flatness we also have $\mathcal{M}' \models_{Y'[\mathcal{F}/\vec{y}]} R\vec{y}$ and thus $\mathcal{M}' \models_{Y'} \exists \vec{y} (R\vec{y} \wedge \varphi'_j)$. Therefore $\mathcal{M}' \models_{\{\emptyset\}[M^k/\vec{u}]} \neg P_j \vec{u} \vee \exists \vec{y} (R\vec{y} \wedge \varphi'_j)$ and thus $\mathcal{M}' \models \bigwedge_{i \leq n} \forall \vec{u} (\neg P_i \vec{u} \vee \exists \vec{y} (R\vec{y} \wedge \varphi'_i))$.

Because $\mathcal{M}' \models_X \varphi'$ and $X(\vec{y}) = R^{\mathcal{M}'}$, by locality $\mathcal{M}' \models \forall \vec{y} (\neg R\vec{y} \vee (R\vec{y} \wedge \varphi'))$. Therefore we can conclude that $\mathcal{M}[X(\vec{y})/R] \models \Phi$.

Suppose then that $\mathcal{M}[X(\vec{y})/R] \models \Phi$. Thus there exist $A_1, \dots, A_n \subseteq M^k$ such that the first order part of Φ holds in $\mathcal{M}' := \mathcal{M}[\vec{A}/\vec{P}, X(\vec{y})/R]$. In particular, $\mathcal{M}' \models \forall \vec{y} (\neg R\vec{y} \vee (R\vec{y} \wedge \varphi'))$ and thus by locality $\mathcal{M}' \models_X \varphi'$.

For the sake of proving the right side of the equivalence of Claim 3, let $j \leq n$ and $\vec{a} \in A_j$. Now $\mathcal{M}' \models \forall \vec{u} (\neg P_j \vec{u} \vee \exists \vec{y} (R\vec{y} \wedge \varphi'_j))$, and thus there are $Y, Y' \subseteq \{\emptyset\}[M^k/\vec{u}]$ such that $Y \cup Y' = \{\emptyset\}[M^k/\vec{u}]$, $\mathcal{M}' \models_Y \neg P_j \vec{u}$ and $\mathcal{M}' \models_{Y'} \exists \vec{y} (R\vec{y} \wedge \varphi'_j)$. Hence there is a function $\mathcal{F} : Y' \rightarrow \mathcal{P}^*(M^m)$ such that $\mathcal{M}' \models_{Y'[\mathcal{F}/\vec{y}]} R\vec{y} \wedge \varphi'_j$.

Let $r := \emptyset[\vec{a}/\vec{u}]$, whence $r \in \{\emptyset\}[M^k/\vec{u}]$. Since $r(\vec{u}) = \vec{a} \in A_j = P_j^{\mathcal{M}'}$ and $\mathcal{M}' \models_Y \neg P_j \vec{u}$, we have $r \notin Y$ and thus $r \in Y'$. Let $\vec{b} \in \mathcal{F}(r)$ and let $s := r[\vec{b}/\vec{y}]$. By flatness, $\mathcal{M}' \models_{\{s\}} R\vec{y}$ and thus $s(\vec{y}) \in R^{\mathcal{M}'} = X(\vec{y})$. Hence there exists $s' \in X$ such that $s'(\vec{y}) = s(\vec{y})$. Since $\mathcal{M}' \models_{\{s\}} \varphi'_j$, by locality also $\mathcal{M}' \models_{\{s'[\vec{a}/\vec{u}]\}} \varphi'_j$. Because $s(\vec{u}) = r(\vec{u}) = \vec{a}$, by Claim 3 we have $\mathcal{M} \models_X \varphi$. \square

Forming a translation from INEX[k] to ESO[k]

The next theorem shows that there is also a translation from INEX[k] to ESO[k]. This translation can be formulated by first eliminating exclusion atoms as in Theorem 4.1 and then inclusion atoms as in Theorem 4.2.

Theorem 4.3. Let $\varphi(\vec{y}) \in \text{INEX}_L[k]$. Now there is an $\text{ESO}_L[k]$ -formula $\Phi(R)$, for which we have

$$\mathcal{M} \models_X \varphi \text{ iff } \mathcal{M}[X(\vec{y})/R] \models \Phi.$$

Proof. Without loss of generality we may assume that each exclusion and inclusion atom in the formula φ is k -ary. We index the exclusion atoms by $(\vec{t}_1 \mid \vec{t}_2)_1, \dots, (\vec{t}_1 \mid \vec{t}_2)_n$. Let P_1, \dots, P_n be k -ary relation variables. Let $\psi \in \text{Sf}(\varphi)$. We define the formula ψ' recursively as follows.

$$\begin{aligned} \psi' &= \psi \text{ if } \psi \text{ is a literal} \\ ((\vec{t}_1 \mid \vec{t}_2)_i)' &= P_i \vec{t}_1 \wedge \neg P_i \vec{t}_2 \text{ for each } i \leq n \\ (\vec{t}_1 \subseteq \vec{t}_2)' &= \vec{t}_1 \subseteq \vec{t}_2 \\ (\psi \wedge \theta)' &= \psi' \wedge \theta', \quad (\psi \vee \theta)' = \psi' \vee \theta' \\ (\exists x \psi)' &= \exists x \psi', \quad (\forall x \psi)' = \forall x \psi'. \end{aligned}$$

We can prove the equivalence of Claim 2 for any $\mu \in \text{Sf}(\varphi)$ by structural induction on μ : Since inclusion atoms are left as they are, their step in the induction is trivial. Other steps can be proven identically as in the proof of Claim 2 within the proof of Theorem 4.1. Thus we have

$$\mathcal{M} \models_X \varphi \text{ iff there exist } A_1, \dots, A_n \subseteq M^k \text{ s.t. } \mathcal{M}[\vec{A}/\vec{P}] \models_X \varphi'.$$

Since φ' contains only inclusion atoms and $\text{Fr}(\varphi) = \text{Fr}(\varphi') = \text{Vr}(\vec{y})$, we can apply Theorem 4.2 for φ' to get an ESO_L -formula $\Psi(R)$ for which we have

$$\mathcal{M} \models_X \varphi' \text{ iff } \mathcal{M}[X(\vec{y})/R] \models \Psi.$$

We can now define $\Phi := \exists P_1 \dots \exists P_n \Psi$, whence Φ is an ESO_L -formula with the free relation variable R . Then we have

$$\begin{aligned} \mathcal{M} \models_X \varphi &\text{ iff there exist } A_1, \dots, A_n \subseteq M^k \text{ s.t. } \mathcal{M}[\vec{A}/\vec{P}] \models_X \varphi' \\ &\text{ iff there exist } A_1, \dots, A_n \subseteq M^k \text{ s.t. } \mathcal{M}[\vec{A}/\vec{P}, X(\vec{y})/R] \models \Psi \\ &\text{ iff } \mathcal{M}[X(\vec{y})/R] \models \Phi. \end{aligned} \quad \square$$

The result of Theorem 4.3 can be formulated equivalently as follows:
All $\text{INEX}[k]$ -definable properties of teams are $\text{ESO}[k]$ -definable.

4.2. Translation from $\text{ESO}[k]$ to $\text{INEX}[k]$

When translating from ESO to INEX , our technique is to simulate second order quantification by replacing the quantifications of k -ary relation variables P simply with quantifications of k -tuples of *first order variables* \vec{w} . The idea is then to choose such values for \vec{w} that in the resulting team X , the relation

$X(\vec{w})$ is the same as the relation that is quantified for the value of P . However, we cannot simulate the quantification of the empty set this way, since the first order variables must be given at least one value. But this problem can be avoided, since any ESO-formula Φ can be written in an equivalent form Φ' which is satisfied if and only if it is satisfied with nonempty interpretations for the quantified relation variables. This is stated in the following easy lemma (for a proof, see [17]).

Lemma 4.4. *Let $\Phi := \exists P_1 \dots \exists P_n \gamma$ be an $\text{ESO}_L[k]$ -formula, where γ is the first order part of Φ . Then there exists $\delta \in \text{ESO}_L[0]$ with the same free relation variables as γ such that the following holds:*

$$\mathcal{M} \models \Phi \text{ iff there exist nonempty } A_1, \dots, A_n \subseteq M^k \text{ s.t. } \mathcal{M}[\vec{A}/\vec{P}] \models \delta.$$

Now we are ready to formulate our translation from $\text{ESO}[k]$ to $\text{INEX}[k]$. For this translation we must require the given teams to be nonempty and assume that the free relation variables in $\text{ESO}_L[k]$ -formulas are at most k -ary. But here we can allow the $\text{ESO}_L[k]$ -formula Φ to have any number of free relation variables instead of just one. Suppose that Φ defines some properties p_1, \dots, p_m for relations R_1, \dots, R_m respectively. Then it is natural to say that $\varphi(\vec{y}_1 \dots \vec{y}_m) \in \text{INEX}_L$ is equivalent with Φ if the relations $X(\vec{y}_1), \dots, X(\vec{y}_m)$ have the properties p_1, \dots, p_m in all teams X in which φ true.

Theorem 4.5. *Let $\Phi(R_1 \dots R_m) \in \text{ESO}_L[k]$, where the free relation variables R_i are at most k -ary. Let $\vec{y}_1, \dots, \vec{y}_m$ be k -tuples of fresh variables. Then there exists an $\text{INEX}_L[k]$ -formula $\varphi(\vec{y}_1 \dots \vec{y}_m)$, such that*

$$\mathcal{M} \models_X \varphi \text{ iff } \mathcal{M}[X(\vec{y}_1)/R_1, \dots, X(\vec{y}_m)/R_m] \models \Phi,$$

for all suitable L -models \mathcal{M} and nonempty teams X .

Proof. Since $\Phi \in \text{ESO}_L[k]$, it is of the form $\Phi = \exists P_1 \dots \exists P_n \gamma$, where P_1, \dots, P_n are relation variables and γ is the first order part of Φ . Without loss of generality, we may assume that $P_1, \dots, P_n, R_1, \dots, R_m$ are all distinct and k -ary. Let δ be the formula given by Lemma 4.4 for the formula γ . Now we have

$$\mathcal{M} \models \Phi \text{ iff there exist nonempty } A_1, \dots, A_n \subseteq M^k \text{ s.t. } \mathcal{M}[\vec{A}/\vec{P}] \models \delta, \quad (\Delta)$$

for all models \mathcal{M} that have interpretations for the relation variables R_1, \dots, R_m . Let $\vec{w}_1, \dots, \vec{w}_n$ be k -tuples of fresh variables. The formula ψ' is defined recur-

sively for each $\psi \in \text{Sf}(\delta)$:

$$\begin{aligned}
\psi' &= \psi \text{ if } \psi \text{ is a literal and neither } P_i \text{ nor } R_j \\
&\quad \text{occurs in } \psi \text{ for any } i \text{ or } j. \\
(P_i \vec{t})' &= \vec{t} \subseteq \vec{w}_i, \quad (\neg P_i \vec{t})' = \vec{t} \mid \vec{w}_i \quad \text{for all } i \leq n \\
(R_i \vec{t})' &= \vec{t} \subseteq \vec{y}_i, \quad (\neg R_i \vec{t})' = \vec{t} \mid \vec{y}_i \quad \text{for all } i \leq m \\
(\psi \wedge \theta)' &= \psi' \wedge \theta' \\
(\psi \vee \theta)' &= \psi' \vee \theta', \quad \text{where } \vee := \bigvee_{\vec{w}_1, \dots, \vec{w}_n, \vec{y}_1, \dots, \vec{y}_m} \\
(\exists x \psi)' &= \exists x \psi', \quad (\forall x \psi)' = \forall x \psi'.
\end{aligned}$$

Now we can define the formula φ simply as:

$$\varphi := \exists \vec{w}_1 \dots \exists \vec{w}_n \delta'.$$

Clearly φ is an $\text{INEX}_L[k]$ -formula and $\text{Fr}(\varphi) = \text{Vr}(\vec{y}_1 \dots \vec{y}_m)^7$. Before proving the claim of this theorem need to prove Claims 4 and 5.

Claim 4. *Let $\mu \in \text{Sf}(\delta)$ and let X be a team such that the variables $\vec{w}_1, \dots, \vec{w}_n, \vec{y}_1, \dots, \vec{y}_m$ are in $\text{dom}(X)$. Let*

$$\mathcal{M}' := \mathcal{M}[X(\vec{w}_1)/P_1, \dots, X(\vec{w}_n)/P_n, X(\vec{y}_1)/R_1, \dots, X(\vec{y}_m)/R_m].$$

Now we have: If $\mathcal{M} \models_X \mu'$, then $\mathcal{M}' \models_X \mu$.

We prove this claim by structural induction on μ :

- If μ is a literal such that neither P_i nor R_j occurs in μ for any $i \leq n$ or $j \leq m$, the claim holds trivially since $\mu' = \mu$.
- Let $\mu = P_j \vec{t}$ for some j (the case $\mu = R_j \vec{t}$ is analogous). Suppose that $\mathcal{M} \models_X (P_j \vec{t})'$, i.e. $\mathcal{M} \models_X \vec{t} \subseteq \vec{w}_j$, and let $s \in X$. Because $\mathcal{M} \models_X \vec{t} \subseteq \vec{w}_j$, there exists $s' \in X$ such that $s'(\vec{w}_j) = s(\vec{t})$. Now we have $s(\vec{t}) \in X(\vec{w}_j) = P_j^{\mathcal{M}'}$, and thus $\mathcal{M}' \models_X P_j \vec{t}$.
- Let $\mu = \neg P_j \vec{t}$ for some j (the case $\mu = \neg R_j \vec{t}$ is analogous). Suppose that $\mathcal{M} \models_X (\neg P_j \vec{t})'$, i.e. $\mathcal{M} \models_X \vec{t} \mid \vec{w}_j$ and let $s \in X$. Since $\mathcal{M} \models_X \vec{t} \mid \vec{w}_j$, we have $s(\vec{t}) \neq s'(\vec{w}_j)$ for each $s' \in X$. Therefore $s(\vec{t}) \notin X(\vec{w}_j) = P_j^{\mathcal{M}'}$, and thus $\mathcal{M}' \models_X \neg P_j \vec{t}$.
- The case $\mu = \psi \wedge \theta$ is straightforward to prove.
- Let $\mu = \psi \vee \theta$. Suppose that $\mathcal{M} \models_X (\psi \vee \theta)'$, i.e. $\mathcal{M} \models_X \psi' \vee \theta'$. Thus there are $Y_1, Y_2 \subseteq X$ s.t. $Y_1 \cup Y_2 = X$, $\mathcal{M} \models_{Y_1} \psi'$ and $\mathcal{M} \models_{Y_2} \theta'$, and if $Y_1, Y_2 \neq \emptyset$,

⁷Also, note that if Φ is an ESO_L -sentence, then φ is an INEX_L -sentence.

then the tuples \vec{w}_i and \vec{y}_j have the same set of values in Y_1 and Y_2 as they have in X (for each $i \leq n$ and $j \leq m$).

If $Y_1 = \emptyset$, then $Y_2 = X$ and thus $\mathcal{M} \models_X \theta'$. By the inductive hypothesis $\mathcal{M}' \models_X \theta$ and thus $\mathcal{M}' \models_X \psi \vee \theta$. Analogously if $Y_2 = \emptyset$, then $\mathcal{M}' \models_X \psi \vee \theta$. Suppose then that $Y_1, Y_2 \neq \emptyset$. Now by the inductive hypothesis we have

$$\begin{cases} \mathcal{M}[Y_1(\vec{w}_i)_{i \leq n}/\vec{P}, Y_1(\vec{y}_j)_{j \leq m}/\vec{R}] \models_{Y_1} \psi \\ \mathcal{M}[Y_2(\vec{w}_i)_{i \leq n}/\vec{P}, Y_2(\vec{y}_j)_{j \leq m}/\vec{R}] \models_{Y_2} \theta. \end{cases}$$

Because \vec{w}_i and \vec{y}_j have the same set of values in Y_1 and Y_2 as in X (for any $i \leq n, j \leq m$), we have $\mathcal{M}' \models_{Y_1} \psi$ and $\mathcal{M}' \models_{Y_2} \theta$. Therefore $\mathcal{M}' \models_X \psi \vee \theta$.

- The cases $\mu = \exists x \psi$ and $\mu = \forall x \psi$ are straightforward to prove.

Claim 5. *Let $\mu \in \text{Sf}(\delta)$ and assume that $A_1, \dots, A_n, B_1, \dots, B_m \subseteq M^k$ are nonempty sets. Let $X \neq \emptyset$ be a team such that $\text{Vr}(\vec{y}_1 \dots \vec{y}_m) \subseteq \text{dom}(X)$ and for each $i \leq m$ and $r \in X \upharpoonright \text{Fr}(\mu)$ the following assumption holds:*

$$X_r(\vec{y}_i) = B_i, \text{ where } X_r := \{s \in X \mid s \upharpoonright \text{Fr}(\mu) = r\}. \quad (\star)$$

This condition can be written equivalently as: For each $r \in X \upharpoonright \text{Fr}(\mu)$, $i \leq m$ and $\vec{b} \in B_i$ there exists $s \in X$ such that $s \upharpoonright (\text{Fr}(\mu) \cup \text{Vr}(\vec{y}_i)) = r[\vec{b}/\vec{y}_i]$. That is, each assignment in $X \upharpoonright \text{Fr}(\varphi)$ can be extended to X with all of the values in B_i .

Now the following implication holds:

$$\text{If } \mathcal{M}' \models_{X \upharpoonright \text{Fr}(\mu)} \mu, \text{ then } \mathcal{M} \models_{X'} \mu',$$

where $\mathcal{M}' := \mathcal{M}[\vec{A}/\vec{P}, \vec{B}/\vec{R}]$ and $X' := X[A_1/\vec{w}_1, \dots, A_n/\vec{w}_n]$.

We prove this claim by structural induction on μ :

- If μ is a literal such that neither P_i nor R_j occurs in μ , then the claim holds by locality since $\mu' = \mu$.
- Let $\mu = P_j \vec{t}$ for some $j \leq n$. Suppose that $\mathcal{M}' \models_{X \upharpoonright \text{Fr}(\mu)} P_j \vec{t}$. Let $s \in X'$ and let $r \in X \upharpoonright \text{Fr}(\mu)$ be an assignment for which $r = s \upharpoonright \text{Fr}(\mu)$. Since $\mathcal{M}' \models_{X \upharpoonright \text{Fr}(\mu)} P_j \vec{t}$, we have $r(\vec{t}) \in P_j^{\mathcal{M}'} = A_j = X'(\vec{w}_j)$. Thus there exists $s' \in X'$ s.t. $s'(\vec{w}_j) = r(\vec{t})$. Now $s(\vec{t}) = r(\vec{t}) = s'(\vec{w}_j)$. Therefore $\mathcal{M} \models_{X'} \vec{t} \subseteq \vec{w}_j$, i.e. $\mathcal{M} \models_{X'} (P_j \vec{t})'$.
- Let $\mu = \neg P_j \vec{t}$ for some $j \leq n$. Suppose that $\mathcal{M}' \models_{X \upharpoonright \text{Fr}(\mu)} \neg P_j \vec{t}$. Let $s, s' \in X'$ and let $r \in X \upharpoonright \text{Fr}(\mu)$ be an assignment s.t. $r = s \upharpoonright \text{Fr}(\mu)$. Because $\mathcal{M}' \models_{X \upharpoonright \text{Fr}(\mu)} \neg P_j \vec{t}$, we have $r(\vec{t}) \notin P_j^{\mathcal{M}'} = A_j = X'(\vec{w}_j)$. Hence it has to be that $r(\vec{t}) \neq s'(\vec{w}_j)$, and thus $s(\vec{t}) = r(\vec{t}) \neq s'(\vec{w}_j)$. Therefore $\mathcal{M} \models_{X'} \vec{t} \mid \vec{w}_j$, i.e. $\mathcal{M} \models_{X'} (\neg P_j \vec{t})'$.

- Let $\mu = R_j \vec{t}$ or $\mu = \neg R_j \vec{t}$ for some $j \leq m$. Note that because the condition (\star) holds for X (with respect to $\text{Fr}(\mu)$), we have $X(\vec{y}_j) = B_j$. Hence $R_j^{\mathcal{M}'} = B_j = X(\vec{y}_j) = X'(\vec{y}_j)$, and thus the cases $\mu = R_j \vec{t}$ and $\mu = \neg R_j \vec{t}$ can be proved analogously as we proved the two previous cases.
- The case $\mu = \psi \wedge \theta$ is straightforward to prove.
- Let $\mu = \psi \vee \theta$. Suppose that $\mathcal{M}' \models_{X \upharpoonright \text{Fr}(\mu)} \psi \vee \theta$, i.e. there are $Y_1^*, Y_2^* \subseteq X \upharpoonright \text{Fr}(\mu)$ such that $Y_1^* \cup Y_2^* = X \upharpoonright \text{Fr}(\mu)$, $\mathcal{M}' \models_{Y_1^*} \psi$ and $\mathcal{M}' \models_{Y_2^*} \theta$. Let

$$Y_1 := \{s \in X \mid s \upharpoonright \text{Fr}(\mu) \in Y_1^*\} \quad \text{and} \quad Y_2 := \{s \in X \mid s \upharpoonright \text{Fr}(\mu) \in Y_2^*\}.$$

Now $Y_1 \upharpoonright \text{Fr}(\mu) = Y_1^*$, $Y_2 \upharpoonright \text{Fr}(\mu) = Y_2^*$ and $Y_1 \cup Y_2 = X$. Let

$$Y_1' := Y_1[A_1/\vec{w}_1, \dots, A_n/\vec{w}_n] \quad \text{and} \quad Y_2' := Y_2[A_1/\vec{w}_1, \dots, A_n/\vec{w}_n].$$

Now $X' = Y_1' \cup Y_2'$. If $Y_1' = \emptyset$, then $Y_2' = X'$ and thus clearly $\mathcal{M} \models_{X'} \psi' \vee \theta'$, i.e. $\mathcal{M} \models_{X'} (\psi \vee \theta)'$. Analogously if $Y_2' = \emptyset$, then $\mathcal{M} \models_{X'} (\psi \vee \theta)'$. Suppose then that $Y_1', Y_2' \neq \emptyset$. Since the condition (\star) holds for X with respect to $\text{Fr}(\mu)$, by the definition of Y_1 it is easy to see that (\star) holds also for Y_1 with respect to $\text{Fr}(\mu)$. Since $\text{Fr}(\psi) \subseteq \text{Fr}(\mu)$, (\star) holds for Y_1 also with respect to $\text{Fr}(\psi)$. Analogously (\star) holds for Y_2 with respect to $\text{Fr}(\theta)$. Therefore, by the inductive hypothesis, $\mathcal{M} \models_{Y_1'} \psi'$ and $\mathcal{M} \models_{Y_2'} \theta'$. We also have $Y_1'(\vec{w}_i) = Y_2'(\vec{w}_i) = A_i = X'(w_i)$ for each $i \leq n$. Furthermore, by the condition (\star) , $Y_1'(\vec{y}_i) = Y_2'(\vec{y}_i) = B_i = X'(y_i)$ for each $i \leq m$. Therefore $\mathcal{M} \models_{X'} \psi' \vee \theta'$, i.e. $\mathcal{M} \models_{X'} (\psi \vee \theta)'$.

- Let $\mu = \exists x \psi$ (the case $\mu = \forall x \psi$ can be proven similarly). Suppose that $\mathcal{M}' \models_{X \upharpoonright \text{Fr}(\mu)} \exists x \psi$. Hence there is $F : X \upharpoonright \text{Fr}(\mu) \rightarrow \mathcal{P}^*(M)$ such that $\mathcal{M}' \models_{(X \upharpoonright \text{Fr}(\mu))[F/x]} \psi$. Let

$$\begin{aligned} G : X &\rightarrow \mathcal{P}^*(M), \quad s \mapsto F(s \upharpoonright \text{Fr}(\mu)) \\ G' : X' &\rightarrow \mathcal{P}^*(M), \quad s \mapsto F(s \upharpoonright \text{Fr}(\mu)). \end{aligned}$$

Now $X[G/x] \upharpoonright \text{Fr}(\psi) = (X \upharpoonright \text{Fr}(\mu))[F/x]$ and therefore $\mathcal{M}' \models_{X[G/x] \upharpoonright \text{Fr}(\psi)} \psi$. Since (\star) holds for X with respect to $\text{Fr}(\mu)$, by the definition of G (\star) holds for $X[G/x]$ with respect to $\text{Fr}(\psi)$. Let $X'' := (X[G/x])[A_1/\vec{w}_1, \dots, A_n/\vec{w}_n]$, whence by the inductive hypothesis we have $\mathcal{M} \models_{X''} \psi'$. By the definition of G' , we have $X'' = X'[G'/x]$, and thus $\mathcal{M} \models_{X'[G'/x]} \psi'$. Hence $\mathcal{M} \models_{X'} \exists x \psi'$, i.e. $\mathcal{M} \models_{X'} (\exists x \psi)'$.

We are now are finally ready prove the claim of this theorem:

$$\mathcal{M} \models_X \varphi \text{ iff } \mathcal{M}[X(\vec{y}_1)/R_1, \dots, X(\vec{y}_m)/R_m] \models \Phi.$$

Suppose first that $\mathcal{M} \models_X \varphi$, i.e. $\mathcal{M} \models_X \exists \vec{w}_1 \dots \exists \vec{w}_n \delta'$. Thus there exist

$$\begin{aligned} \mathcal{F}_1 &: X \rightarrow \mathcal{P}^*(M^k) \\ \mathcal{F}_2 &: X[\mathcal{F}_1/\vec{w}_1] \rightarrow \mathcal{P}^*(M^k) \\ &\vdots \\ \mathcal{F}_n &: X[\mathcal{F}_1/\vec{w}_1, \dots, \mathcal{F}_{n-1}/\vec{w}_{n-1}] \rightarrow \mathcal{P}^*(M^k) \\ \text{s.t. } &\mathcal{M} \models_{X'} \delta', \text{ where } X' := X[\mathcal{F}_1/\vec{w}_1, \dots, \mathcal{F}_n/\vec{w}_n]. \end{aligned}$$

Let $\mathcal{M}' := \mathcal{M}[X'(\vec{w}_1)/P_1, \dots, X'(\vec{w}_n)/P_n, X'(\vec{y}_1)/R_1, \dots, X'(\vec{y}_m)/R_m]$. Now by Claim 4, we have $\mathcal{M}' \models_{X'} \delta$. Because $X' \upharpoonright \text{Fr}(\delta) = \{\emptyset\}$, by locality $\mathcal{M}' \models \delta$. Since $X'(\vec{y}_i) = X(\vec{y}_i)$ for each $i \leq m$, we have $\mathcal{M}[X(\vec{y}_1)/R_1, \dots, X(\vec{y}_m)/R_m] \models \Phi$.

Suppose then that $\mathcal{M}[X(\vec{y}_1)/R_1, \dots, X(\vec{y}_m)/R_m] \models \Phi$. Thus, by the equation (Δ) , there are nonempty sets $A_1, \dots, A_n \subseteq M^k$ such that

$$\mathcal{M}' \models \delta, \text{ where } \mathcal{M}' := \mathcal{M}[A_1/P_1, \dots, A_n/P_n, X(\vec{y}_1)/R_1, \dots, X(\vec{y}_m)/R_m].$$

Since, by the assumptions, $X \neq \emptyset$ and $\text{Fr}(\vec{y}_1 \dots \vec{y}_m) \subseteq \text{dom}(X)$, we have $X(\vec{y}_i) \neq \emptyset$ for each $i \leq m$. We define the function \mathcal{F}_i for each $i \leq n$ by

$$\mathcal{F}_i : X[\mathcal{F}_1/\vec{w}_1, \dots, \mathcal{F}_{i-1}/\vec{w}_{i-1}] \rightarrow \mathcal{P}^*(M^k), \quad s \mapsto A_i.$$

Let $X' := X[\mathcal{F}_1/\vec{w}_1, \dots, \mathcal{F}_n/\vec{w}_n]$, whence $X' = X[A_1/\vec{w}_1, \dots, A_n/\vec{w}_n]$. Since $X \upharpoonright \text{Fr}(\delta) = \{\emptyset\}$, the condition (\star) in Claim 5 holds for the team X with respect to $\text{Fr}(\delta)$. We also have $\mathcal{M}' \models_{X \upharpoonright \text{Fr}(\delta)} \delta$ and thus by Claim 5 we obtain $\mathcal{M} \models_{X'} \delta'$. Hence $\mathcal{M} \models_X \exists \vec{w}_1 \dots \exists \vec{w}_n \delta'$, i.e. $\mathcal{M} \models_X \varphi$. \square

Remark. By Theorem 4.5, for each $\text{ESO}_L[k]$ -formula $\Phi(R)$, for which R is at most k -ary, there exists an $\text{INEX}_L[k]$ -formula $\varphi(\vec{y})$ such that for all $X \neq \emptyset$:

$$\mathcal{M} \models_X \varphi \text{ iff } \mathcal{M}[X(\vec{y})/R] \models \Phi.$$

Without the requirement of non-empty teams and the arity restriction on R , this would be the converse of Theorem 4.3. But due empty team property of INEX, the left side of the equivalence is always true for the empty team and any formula of INEX. Thus, when defining classes of relations with INEX, we can only define such classes that include the empty relation. The arity restriction is also necessary since it can be shown that for any k there are $\text{ESO}[k]$ -definable properties of $(k+1)$ -ary relations $X(\vec{y})$ that cannot be defined in $\text{INEX}[k]$. A proof for this claim will be presented in a future work by the author.

Since Theorem 4.3 and Theorem 4.5 can also be proven for $\text{INEX}_L[k]$ - and $\text{ESO}_L[k]$ -sentences, we obtain the following corollary.

Corollary 4.6. *On the level of sentences $\text{INEX}[k]$ captures the expressive power of $\text{ESO}[k]$. In particular, $\text{INEX}[1]$ captures EMSO .*

As a direct corollary we also obtain a strict arity hierarchy for INEX , since the arity hierarchy for ESO (with arbitrary vocabulary) is strict, as shown by Ajtai [1] in 1983. As mentioned in the introduction, k -ary dependence and independence logics capture the fragment of ESO where at most $(k-1)$ -ary functions can be quantified. This fragment differs from $\text{ESO}[k]$ at least when k is one or two – and presumably for any k . Hence it appears that $\text{INEX}[k]$ does not correspond to l -ary independence logic for any k and l , even though without arity bounds these two logics are equivalent.

On the duality of inclusion and exclusion atoms

For the last topic in this section, we will discuss the relationship of inclusion and exclusion atoms. We will also consider natural candidates for the semantics of negated inclusion and exclusion atoms. In our translation in Theorem 4.5 we used inclusion and exclusion atoms in a dualistic way by replacing atomic formulas $P\vec{t}$ with inclusion atoms and negated atomic formulas $\neg P\vec{t}$ with exclusion atoms. This correspondence becomes more obvious when we reformulate the truth conditions for $P\vec{t}$ and $\neg P\vec{t}$ (compare with Definition 2.1) as follows:

$$\mathcal{M} \models_X P\vec{t} \text{ iff } X(\vec{t}) \subseteq P^{\mathcal{M}} \quad \text{and} \quad \mathcal{M} \models_X \neg P\vec{t} \text{ iff } X(\vec{t}) \subseteq \overline{P^{\mathcal{M}}}.$$

The truth conditions for inclusion and exclusion atoms can be written in a form that is very similar to the equivalences above:

$$\mathcal{M} \models_X \vec{t}_1 \subseteq \vec{t}_2 \text{ iff } X(\vec{t}_1) \subseteq X(\vec{t}_2) \quad \text{and} \quad \mathcal{M} \models_X \vec{t}_1 | \vec{t}_2 \text{ iff } X(\vec{t}_1) \subseteq \overline{X(\vec{t}_2)}.$$

As we argued earlier (Observation 2.1), the semantics of a contradictory negation ($\mathcal{M} \models_X \neg\varphi$ iff $\mathcal{M} \not\models_X \varphi$) is not a very natural choice of semantics for the negated atoms. Instead, it would be more natural to have such a semantics that is similar to the semantics of literals. From this viewpoint, a natural candidate for a semantics of a negated inclusion atom would be the following:

$$\mathcal{M} \models_X \neg(\vec{t}_1 \subseteq \vec{t}_2) \text{ iff } X(\vec{t}_1) \subseteq \overline{X(\vec{t}_2)}. \quad (\neg \subseteq)$$

Then we would have $\neg(\vec{t}_1 \subseteq \vec{t}_2) \equiv \vec{t}_1 | \vec{t}_2$. Therefore, if we allow the use of negated atoms in $\text{INC}[k]$ with our semantics, the resulting logic is equivalent with $\text{INEX}[k]$. Note that since the exclusion relation is symmetric, our choice of semantics leads to the following equivalence:

$$\neg(\vec{t}_1 \subseteq \vec{t}_2) \equiv \vec{t}_1 | \vec{t}_2 \equiv \vec{t}_2 | \vec{t}_1 \equiv \neg(\vec{t}_2 \subseteq \vec{t}_1).$$

Hence, by this definition, $\neg(\vec{t}_1 \subseteq \vec{t}_2) \equiv \neg(\vec{t}_2 \subseteq \vec{t}_1)$ even though $\vec{t}_1 \subseteq \vec{t}_2 \not\equiv \vec{t}_2 \subseteq \vec{t}_1$. This kind of property of a negated atom might be a bit exotic, but not unthinkable, since our negation is not a contradictory negation.

Let us then consider semantics for the negated exclusion atom $\neg(\vec{t}_1 \mid \vec{t}_2)$. Semantics of the inclusion atom $\vec{t}_1 \subseteq \vec{t}_2$ is not a possible choice here, since by the symmetry of the exclusion relation, we must have $\neg(\vec{t}_1 \mid \vec{t}_2) \equiv \neg(\vec{t}_2 \mid \vec{t}_1)$. The truth condition $\mathcal{M} \models_X \vec{t}_1 \mid \vec{t}_2$ iff $X(\vec{t}_1) \cap X(\vec{t}_2) = \emptyset$, naturally gives us the following candidate for a semantics.

$$\mathcal{M} \models_X \neg(\vec{t}_1 \mid \vec{t}_2) \text{ iff } X(\vec{t}_1) = X(\vec{t}_2) \quad (\neg \mid)$$

Now we have $\neg(\vec{t}_1 \mid \vec{t}_2) \equiv \neg(\vec{t}_2 \mid \vec{t}_1)$, as required, and $\neg(\vec{t}_1 \mid \vec{t}_2) \equiv \vec{t}_1 \subseteq \vec{t}_2 \wedge \vec{t}_2 \subseteq \vec{t}_1$. This choice of semantics is actually equivalent with the semantics of *equiextension atom* $\vec{t}_1 \bowtie \vec{t}_2$ that was introduced by Galliani in [4]. This atom has been shown equivalent with the inclusion atom $\vec{t}_1 \subseteq \vec{t}_2$ of the same arity ([4]). Hence if we allow the use of negated atoms in EXC[k] with our semantics, the resulting logic turns out to be equivalent with INEX[k].

With our choices for semantics of negated inclusion and exclusion atoms, $(\neg \subseteq)$ and $(\neg \mid)$, we have $\neg(\vec{t}_1 \subseteq \vec{t}_2) \equiv \vec{t}_1 \mid \vec{t}_2$ and $\neg(\vec{t}_1 \mid \vec{t}_2) \equiv \vec{t}_1 \bowtie \vec{t}_2$. Now the exclusion atom is equivalent with the negated inclusion atom, but not vice versa. However, the negated exclusion atom is equally expressive as the inclusion atom of the corresponding arity. Hence even though inclusion and exclusion atoms are not exactly negations of each other, they nevertheless have a dualistic relationship. We could extend FO with either of these atoms and allow the use of its negation to obtain a logic equivalent to INEX.

In team semantics we must require all formulas to be in negation normal form. This can be seen as one of the weaknesses of this framework since the free use of negation is natural for a logic. Dependence logic and other related logics have also been criticized for not having sensible semantics for negated atoms⁸. However, there has not been much research on these issues.

In order to solve these problems, Kuusisto [16] has presented an alternative framework called *double team semantics*. In this approach there are always two teams – a “verifying team” and a “falsifying team”. This allows to use negations freely, whence it just swaps the roles of these two teams. This approach has received relatively little attention, but we believe that it should be studied further in order to understand the role of negation in team semantics more deeply.

⁸Originally, in [18], negations were allowed to appear in front of dependence atoms. But the semantics for negated dependence atom was defined such that it was true only in the empty team. This way both empty team property and downwards closure were preserved.

5. Examples of some INEX-definable properties

In this section we present several examples on the expressibility of inclusion-exclusion logic. Within these examples we also utilize several of the new operators we introduced in Section 3. Although all of the properties expressed here are known to be expressible in INEX by the results of the previous section, we believe that these examples are valuable for demonstrating the nature of inclusion-exclusion logic and team semantics in general.

By Corollary 4.6 we know that, in particular, all EMSO-definable properties of models can be expressed by using only unary inclusion and exclusion atoms. In the next example we show how two classical EMSO-definable properties of graphs can be defined in INEX[1].

Example 5.1. Let $\mathcal{G} = (V, E)$ be an undirected graph. Then we have

(a) \mathcal{G} is disconnected if and only if

$$\mathcal{G} \models \exists x_1 \exists x_2 \left(x_1 \mid x_2 \wedge \forall z (z \subseteq x_1 \vee z \subseteq x_2) \wedge (\forall y_1 \subseteq x_1)(\forall y_2 \subseteq x_2) \neg E y_1 y_2 \right).$$

(b) \mathcal{G} is k -colorable if and only if

$$\begin{aligned} \mathcal{G} \models \gamma_{\leq k} \vee \exists x_1 \dots \exists x_k \left(\bigwedge_{i \neq j} x_i \mid x_j \wedge \forall z \left(\bigvee_{i \leq k} z \subseteq x_i \right) \right. \\ \left. \wedge \bigwedge_{i \leq k} (\forall y_1 \subseteq x_i)(\forall y_2 \subseteq x_i) \neg E y_1 y_2 \right), \end{aligned}$$

where $\gamma_{\leq k} := \exists x_1 \dots \exists x_k \forall y (\bigvee_{i \leq k} y = x_i)$.

We explain briefly why these equivalences hold: In (a) we first quantify two nonempty sets for the values of x_1 and x_2 . We use exclusion atom to guarantee that these sets are disjoint. The formula $\forall z (z \subseteq x_1 \vee z \subseteq x_2)$ checks that the union of these sets covers the whole set of vertices (recall Example 2.1). Finally we use universal inclusion quantifiers to confirm that for any pair of elements chosen within these sets, there is no edge between them.

In (b) we first check if $|V| \leq k$, in which case the graph would be trivially k -colorable. If that is not the case, we can quantify k nonempty disjoint sets which represent the coloring of the graph. Confirming that these sets are disjoint and cover all the vertices can be done similarly as in (a). Finally we confirm that the coloring is correct by choosing any pair of vertices within an unicolored set and checking that there is no edge between them.

The properties in Example 5.1 could also be expressed in EMSO and then we could directly use our translation in Theorem 4.5 to express these properties in INEX[1]. This method would give us sentences that are only slightly longer

than the ones we have given above. However, the sentences above are not only shorter but also demonstrate the usefulness of universal inclusion quantifier.

In the next example we demonstrate how we can use our translation in Theorem 4.5 to apply techniques of ESO directly to inclusion-exclusion logic. In $\text{ESO}[k+1]$ we can quantify a k -ary function by quantifying a $(k+1)$ -ary relation and giving requirements that it is a function. We can do this analogously in inclusion-exclusion logic; see the following example.

Example 5.2. Let φ be an $\text{INEX}_{L \cup \{F\}}$ -sentence where F is a $(k+1)$ -ary relation symbol. Let \vec{x} be a $(k+1)$ -tuple of fresh variables. The formula $\exists F \varphi$, where F is quantified as a k -ary function, is equivalent with the INEX_L -sentence:

$$\xi := \exists \vec{x} \left(\psi_1(\vec{x}) \wedge \psi_2(\vec{x}) \wedge \varphi' \right), \text{ where}$$

$$\begin{cases} \psi_1(\vec{x}) := \forall \vec{y} \exists z (\vec{y}z \subseteq \vec{x}) \\ \psi_2(\vec{x}) := \forall \vec{y} \forall z_1 \forall z_2 \left((\vec{y}z_1 \mid \vec{x} \vee_{\vec{x}} \vec{y}z_2 \mid \vec{x}) \vee_{\vec{x}} z_1 = z_2 \right) \end{cases}$$

and φ' is a formula obtained from φ by replacing all subformulas of the form $F\vec{t}$ with inclusion atoms $\vec{t} \subseteq \vec{x}$, formulas $\neg F\vec{t}$ with exclusion atoms $\vec{t} \mid \vec{x}$ and all disjunctions with the disjunctions that preserve the values of the tuple \vec{x} .

Note that ψ_1 and ψ_2 above are derived by changing the corresponding $\text{ESO}_{L \cup \{F\}}$ -sentences $\forall \vec{y} \exists z F\vec{y}z$ and $\forall \vec{y} \forall z_1 \forall z_2 ((F\vec{y}z_1 \wedge F\vec{y}z_2) \rightarrow z_1 = z_2)$ to negation normal form and then directly using our translation (Theorem 4.5) from ESO to INEX.

If we also want the quantified function to injective or surjective, we can add either of the following formulas inside the brackets of the formula ξ above:

$$\begin{aligned} \psi_{\text{inj}}(\vec{x}) &:= \forall \vec{y}_1 \forall \vec{y}_2 \forall z \left((\vec{y}_1z \mid \vec{x} \vee_{\vec{x}} \vec{y}_2z \mid \vec{x}) \vee_{\vec{x}} \vec{y}_1 = \vec{y}_2 \right) \\ \psi_{\text{surj}}(\vec{x}) &:= \forall z \exists \vec{y} (\vec{y}z \subseteq \vec{x}). \end{aligned}$$

By using the method of the previous example, we can define infinity of a model in $\text{INEX}[2]$ by simply saying that we can existentially quantify the values of variables x_1 and x_2 in such a way that in the resulting team X the relation $X(x_1x_2)$ is a function that is injective but not surjective.

Example 5.3. Let $\delta_{\text{inf}} \in \text{INEX}_L[2]$ such that

$$\delta_{\text{inf}} := \exists x_1x_2 \left(\psi_1(x_1x_2) \wedge \psi_2(x_1x_2) \wedge \psi_{\text{inj}}(x_1x_2) \wedge \exists z \forall y (yz \mid x_1x_2) \right),$$

where the formulas ψ_1 , ψ_2 and ψ_{inj} are as in the previous example. Now a model \mathcal{M} is infinite if and only if $\mathcal{M} \models \delta_{\text{inf}}$. Note that this property cannot be expressed by using only unary atoms since it is not EMSO-definable.

The expressive power of $\text{INEX}[2]$ is rather strong also on the level of formulas

since, by Theorem 4.5, all ESO[2]-definable properties of 2-ary relations in teams are definable in INEX[2]. In particular, we can say in INEX[2] that a certain variable y gets infinitely many values within a team. This can be done simply by relativizing (recall Subsection 3.5) the sentence δ_{inf} , of the previous example, on the variable y . See the following example.

Example 5.4. Let δ_{inf} be as above and let X be a nonempty team such that the variables in δ_{inf} are not in $\text{dom}(X)$. Now for every $y \in \text{dom}(X)$ we have

$$\mathcal{M} \models_X \delta_{\text{inf}} \upharpoonright y \text{ iff } X(y) \text{ is infinite.}$$

If the team X has a finite domain, we can now say that X consists of infinitely many assignments. That is, if $\text{dom}(X) = \{y_1, \dots, y_n\}$, then we have

$$\mathcal{M} \models_X \bigsqcup_{i \leq n} (\delta_{\text{inf}} \upharpoonright y_i) \text{ iff } X \text{ is infinite.}$$

Because of locality property, there cannot be any single INEX_L -formula that would define the infinity of a team with arbitrary domain. For the same reason we cannot define this property in INEX for teams with infinite domain (even if it is fixed). To see this, consider a team X for which $\text{dom}(X) = \{x_i \mid i \in \mathbb{N}\}$ and $\{s(x_0 x_1 x_2 \dots) \mid s \in X\} = \{0, 1\}^{\mathbb{N}}$. Now X is infinite but $X \upharpoonright V$ is finite for every finite $V \subseteq \text{dom}(X)$. But if $\varphi \in \text{INEX}_L$ such that $\text{Fr}(\varphi) \subseteq \text{dom}(X)$, by locality φ is true in X if and only if it is true in $X \upharpoonright \text{Fr}(\varphi)$. Note that these restrictions hold for any logics with locality property – such as dependence and independence logics. Therefore if our logic is local, we can only define infinity of teams that have a fixed finite domain.

Since infinity of a team is not downwards closed property, it cannot be expressed in dependence logic. By the results of Galliani [4] we know that it is expressible in independence logic. However, to our best knowledge, nobody has presented an explicit formula that would define this property in independence logic (or any other logic with team semantics). If we would express this property in independence logic by directly using the translations given by Galliani [4], the corresponding formula would be very complicated.

In Example 5.4 we defined infinity of a team with a rather simple formula that was constructed in an intuitive way by using methods introduced in this paper. This was just one particular example, but we hope that this demonstrates how our work for this framework can be useful for deriving concrete formulas defining desired properties of models or teams.

6. Conclusion

In this paper we have studied the expressive power of inclusion and exclusion atoms. These two simple types of atoms make a natural pair by having a

dualistic relationship. Our main topic of interest was how does the arity of these atoms affect their expressive power. We showed that INEX has a strict arity hierarchy, and when restricted to $\text{INEX}[k]$, there is a natural connection to $\text{ESO}[k]$ on the level of both sentences and formulas.

When translating from ESO to INEX, atomic k -ary relations translate naturally into k -ary inclusion atoms and analogously negated atomic k -ary relations translate into k -ary exclusion atoms. This simple correspondence is somewhat surprising considering how different these two logics seem by first glance. It is also interesting how in team semantics we can use quantified k -tuples of variables to simulate quantified k -ary relation variables. That is, we can “embed” second order quantification within the standard first order quantification.

Even though INEX is equivalent with independence logic in general, it turned out that the relationship is not so clear when restricting the arities of atoms. Despite being closely related, these logics are of different nature. It appears that inclusion and exclusion atoms are naturally connected with relations while dependence and independence atoms are with functions.

The translations we used between $\text{INEX}[k]$ and ESO are very different from the ones used between k -ary dependence and independence logics and ESO on the level of sentences ([2, 5]). The methods in our proofs also differ from Galiani’s translations ([4]) between INEX-formulas and ESO-formulas (without any arity restriction). These earlier translations are not compositional in the sense that they work only for ESO-formulas in a special normal form. Our translations are all compositional and, particularly the ones in Theorems 4.1 and 4.5, very natural and do not increase the size of the formulas significantly.

In the translation from $\text{ESO}[k]$ to $\text{INEX}[k]$, term value preserving disjunction played an important role. This is a useful operator for any logic with team semantics, since the splitting of the team when evaluating disjunctions tends to lose information. This operator has several natural variants that would be interesting to be studied further either independently or in by adding them to some other logic with team semantics – such as dependence logic.

We also introduced natural semantics for inclusion and exclusion quantifiers and defined them in INEX. With these quantifiers we can restrict the range of quantification to certain sets of values within a team. One practical application of this was the relativization of formulas. *Existential* inclusion and exclusion quantifiers turned out to be equivalent with inclusion and exclusion atoms, and this naturally lead to the definition of inclusion and exclusion friendly logics. However, as discussed in Subsection 3.3, properties of *universal* inclusion and exclusion quantifiers are not so clear, and there still some open questions.

By our results on formulas, we know that all $\text{ESO}[k]$ -definable properties of at most k -ary relations (in teams) can be defined in $\text{INEX}[k]$, and that $\text{ESO}[k]$ is the upper bound for the expressive power of $\text{INEX}[k]$. But these limits are not strict and when the arity of relations gets higher than the arity of atoms,

things get quite interesting. In a future work we will pursue this topic further by showing, e.g, that for 2-ary relations there are some very simple ESO[0]-definable properties which are not INEX[1]-definable, but there are also some quite complex INEX[1]-definable properties which are not ESO[0]-definable.

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Appendix: Proof for claim 3

In this appendix we will present the proof for Claim 3 that was used in the translation from $\text{INC}[k]$ to $\text{ESO}[k]$. With the assumptions of Theorem 4.2, the following equivalence holds for all $\mu \in \text{Sf}(\varphi)$ and all suitable teams X :

$$\begin{aligned} \mathcal{M} \models_X \mu \quad \text{iff} \quad & \text{there exist } A_1, \dots, A_n \subseteq M^k \text{ s.t. } \mathcal{M}[\vec{A}/\vec{P}] \models_X \mu', \\ & \text{and for any } i \leq n \text{ and tuple of elements } \vec{a} \in A_i \\ & \text{there exists } s \in X \text{ s.t. } \mathcal{M}[\vec{A}/\vec{P}] \models_{\{s[\vec{a}/\vec{u}]\}} \mu'_i. \end{aligned}$$

We first need to present two more claims. The first one is about the trivial cases when the inclusion atom $(\vec{t}_1 \subseteq \vec{t}_2)_i$ does not occur in a formula $\mu \in \text{Sf}(\varphi)$.

Claim I. *Let $\mu \in \text{Sf}(\varphi)$ and assume that $i \leq n$ is an index such that the atom $(\vec{t}_1 \subseteq \vec{t}_2)_i$ does not occur in μ . Then the following equivalences hold:*

$$\mathcal{M} \models_X \mu' \quad \text{iff} \quad \mathcal{M} \models_X \mu'_i \quad \text{iff} \quad \mathcal{M} \models_{\{s[\vec{a}/\vec{u}]\}} \mu'_i \text{ for all } \vec{a} \in M^k \text{ and } s \in X.$$

Proof. The first equivalence can be proved by a simple induction on μ . Since $(\vec{t}_1 \subseteq \vec{t}_2)_i$ does not occur in μ , the definitions of μ' and μ'_i differ only when $\mu = \forall x \psi$. But then $\mu' = \forall x \psi'$ and $\mu'_i = \exists x \psi'_i \wedge \forall x \psi'$ are equivalent, by assuming that $\psi' \equiv \psi'_i$ by the inductive hypothesis. For the second equivalence, we note that since $(\vec{t}_1 \subseteq \vec{t}_2)_i$ does not occur in μ , none of the variables in \vec{u} occurs in μ'_i . Hence the second equivalence holds by locality and flatness. \square

The second claim we need shows that we can always extend a team that satisfies the formula μ' with any teams that satisfy μ'_i for some $i \leq n$.

Claim II. *Let $\mu \in \text{Sf}(\varphi)$, $i \leq n$ and let X_1, X_2 be teams for which it holds that $\text{dom}(X_2) = \text{dom}(X_1) \cup \text{Vr}(\vec{u})$. Then the following implication holds:*

If $\mathcal{M} \models_{X_1} \mu'$ and $\mathcal{M} \models_{X_2} \mu'_i$, then $\mathcal{M} \models_{X_1 \cup X_2^} \mu'$, where $X_2^* := X_2 \upharpoonright \text{dom}(X_1)$.*

Proof. We prove this claim by structural induction on μ :

- Let μ be a literal and suppose that $\mathcal{M} \models_{X_1} \mu'$ and $\mathcal{M} \models_{X_2} \mu'_i$. Now by locality $\mathcal{M} \models_{X_2^*} \mu'_i$ and furthermore by flatness $\mathcal{M} \models_{X_1 \cup X_2^*} \mu'$.
- Let $\mu = (\vec{t}_1 \subseteq \vec{t}_2)_j$ for some $j \leq n$. Suppose that $\mathcal{M} \models_{X_1} ((\vec{t}_1 \subseteq \vec{t}_2)_j)'$ and $\mathcal{M} \models_{X_2} ((\vec{t}_1 \subseteq \vec{t}_2)_j)_i'$. By the first assumption, $\mathcal{M} \models_{X_1} P_j \vec{t}_1$. If $j \neq i$, then $\mathcal{M} \models_{X_2} P_j \vec{t}_1$, and if $j = i$, then $\mathcal{M} \models_{X_2} (\vec{u} = \vec{t}_2) \wedge P_j \vec{t}_1$. Thus in either case we have $\mathcal{M} \models_{X_2} P_j \vec{t}_1$. Since none of the variables in \vec{u} occurs in $\text{Vr}(\vec{t}_1)$, by locality $\mathcal{M} \models_{X_2^*} P_j \vec{t}_1$. Because $\mathcal{M} \models_{X_1} P_j \vec{t}_1$ and $\mathcal{M} \models_{X_2^*} P_j \vec{t}_1$, by flatness we have $\mathcal{M} \models_{X_1 \cup X_2^*} P_j \vec{t}_1$. That is, $\mathcal{M} \models_{X_1 \cup X_2^*} ((\vec{t}_1 \subseteq \vec{t}_2)_j)'$.
- The case $\mu = \psi \wedge \theta$ is straightforward to prove.
- Let $\mu = \psi \vee \theta$. Suppose that $\mathcal{M} \models_{X_1} (\psi \vee \theta)'$ and $\mathcal{M} \models_{X_2} (\psi \vee \theta)_i'$. By the first assumption, $\mathcal{M} \models_{X_1} \psi' \vee \theta'$, i.e. there are $Y_1, Y_1' \subseteq X_1$ s.t. $Y_1 \cup Y_1' = X_1$, $\mathcal{M} \models_{Y_1} \psi'$ and $\mathcal{M} \models_{Y_1'} \theta'$.

Suppose first that $(\vec{t}_1 \subseteq \vec{t}_2)_i$ occurs in ψ . Because then $(\psi \vee \theta)_i' = \psi'_i$, we have $\mathcal{M} \models_{X_2} \psi'_i$, and thus by the inductive hypothesis $\mathcal{M} \models_{Y_1 \cup X_2^*} \psi'$. Now $(Y_1 \cup X_2^*) \cup Y_1' = (Y_1 \cup Y_1') \cup X_2^* = X_1 \cup X_2^*$. Hence $\mathcal{M} \models_{X_1 \cup X_2^*} \psi' \vee \theta'$, i.e. $\mathcal{M} \models_{X_1 \cup X_2^*} (\psi \vee \theta)'$. The case when $(\vec{t}_1 \subseteq \vec{t}_2)_i$ occurs in θ is analogous.

Suppose then that $(\vec{t}_1 \subseteq \vec{t}_2)_i$ does not occur in $\psi \vee \theta$. Then $\mathcal{M} \models_{X_2} \psi'_i \vee \theta'_i$, i.e. there exist $Y_2, Y_2' \subseteq X_2$ s.t. $Y_2 \cup Y_2' = X_2$, $\mathcal{M} \models_{Y_2} \psi'_i$ and $\mathcal{M} \models_{Y_2'} \theta'_i$. By the inductive hypothesis, $\mathcal{M} \models_{Y_1 \cup Y_2^*} \psi'$ and $\mathcal{M} \models_{Y_1' \cup Y_2'^*} \theta'$. Now

$$\begin{aligned} (Y_1 \cup Y_2^*) \cup (Y_1' \cup Y_2'^*) &= (Y_1 \cup Y_1') \cup (Y_2^* \cup Y_2'^*) \\ &= (Y_1 \cup Y_1') \cup (Y_2 \cup Y_2')^* = X_1 \cup X_2^*. \end{aligned}$$

Hence $\mathcal{M} \models_{X_1 \cup X_2^*} \psi' \vee \theta'$, i.e. $\mathcal{M} \models_{X_1 \cup X_2^*} (\psi \vee \theta)'$.

- Let $\mu = \exists x \psi$. Suppose that $\mathcal{M} \models_{X_1} (\exists x \psi)'$ and $\mathcal{M} \models_{X_2} (\exists x \psi)_i'$. Hence $\mathcal{M} \models_{X_1} \exists x \psi'$ and $\mathcal{M} \models_{X_2} \exists x \psi'_i$. Hence there are $F_1 : X_1 \rightarrow \mathcal{P}^*(M)$ and $F_2 : X_2 \rightarrow \mathcal{P}^*(M)$ s.t. $\mathcal{M} \models_{X_1[F_1/x]} \psi'$ and $\mathcal{M} \models_{X_2[F_2/x]} \psi'_i$. By the inductive hypothesis $\mathcal{M} \models_{X_1[F_1/x] \cup (X_2[F_2/x])^*} \psi'$. Let

$$F_2^* : X_2^* \rightarrow \mathcal{P}^*(M), \quad s \mapsto \left\{ b \in F_2(s[\vec{a}/\vec{u}]) \mid s[\vec{a}/\vec{u}] \in X_2, \vec{a} \in M^k \right\}$$

$$F : X_1 \cup X_2^* \rightarrow \mathcal{P}^*(M), \quad \begin{cases} s \mapsto F_1(s) & \text{if } s \in X_1 \setminus X_2^* \\ s \mapsto F_2^*(s) & \text{if } s \in X_2^* \setminus X_1 \\ s \mapsto F_1(s) \cup F_2^*(s) & \text{if } s \in X_1 \cap X_2^*. \end{cases}$$

By the definitions of F_2^* and F , we have

$$X_1[F_1/x] \cup (X_2[F_2/x])^* = X_1[F_1/x] \cup X_2^*[F_2^*/x] = (X_1 \cup X_2^*)[F/x].$$

Hence $\mathcal{M} \models_{X_1 \cup X_2^*} \exists x \psi'$, i.e. $\mathcal{M} \models_{X_1 \cup X_2^*} (\exists x \psi)'$.

- Let $\mu = \forall x \psi$. Suppose $\mathcal{M} \models_{X_1} (\forall x \psi)'$ and $\mathcal{M} \models_{X_2} (\forall x \psi)'_i$. Thus $\mathcal{M} \models_{X_1} \forall x \psi'$ and $\mathcal{M} \models_{X_2} \exists x \psi'_i \wedge \forall x \psi'$. Since $\mathcal{M} \models_{X_2} \forall x \psi'$, by locality $\mathcal{M} \models_{X_2^*} \forall x \psi'$. Thus by flatness $\mathcal{M} \models_{X_1 \cup X_2^*} \forall x \psi'$, i.e. $\mathcal{M} \models_{X_1 \cup X_2^*} (\forall x \psi)'$. \square

Now we are finally ready to prove Claim 3:

$$\begin{aligned} \mathcal{M} \models_X \mu \quad \text{iff} \quad & \text{there exist } A_1, \dots, A_n \subseteq M^k \text{ s.t. } \mathcal{M}' \models_X \mu', \\ & \text{and for any } i \leq n \text{ and } \vec{a} \in A_i \text{ there is } s \in X \text{ s.t. } \mathcal{M}' \models_{\{s[\vec{a}/\vec{u}]\}} \mu'_i, \end{aligned}$$

where $\mathcal{M}' := \mathcal{M}[\vec{A}/\vec{P}]$.

Proof. We first examine the special case when $X = \emptyset$: For the other direction of the equivalence, suppose that $\mathcal{M} \models_X \mu$. Let $A_i := \emptyset$ for each $i \leq n$ and let $\mathcal{M}' := \mathcal{M}[\vec{A}/\vec{P}]$. Because $X = \emptyset$, we have $\mathcal{M}' \models_X \mu'$, and since $A_i = \emptyset$ for each $i \leq n$, the rest of the right side of the equivalence holds trivially. The other direction is clear since $\mathcal{M} \models_{\emptyset} \mu$ holds always. We may thus assume that $X \neq \emptyset$.

- If μ is a literal, the claim holds trivially (we can choose $A_i := \emptyset$ for each $i \leq n$ when proving the other direction of the equivalence).
- Let $\mu = (\vec{t}_1 \subseteq \vec{t}_2)_j$ for some $j \leq n$. Suppose first that $\mathcal{M} \models_X \vec{t}_1 \subseteq \vec{t}_2$. Let

$$\mathcal{M}' := \mathcal{M}[\vec{A}/\vec{P}], \text{ where } A_i := \begin{cases} X(\vec{t}_1) & \text{if } i = j \\ \emptyset & \text{else.} \end{cases}$$

Since $X(\vec{t}_1) = A_j = P_j^{\mathcal{M}'}$, we have $\mathcal{M}' \models_X P_j \vec{t}_1$, i.e. $\mathcal{M}' \models_X (\vec{t}_1 \subseteq \vec{t}_2)'$.

Let $i \in \{1, \dots, n\} \setminus \{j\}$ and let $\vec{a} \in A_i$. Since $\mathcal{M}' \models_X P_j \vec{t}_1$ we can choose any $s \in X$ ($\neq \emptyset$), and then by flatness $\mathcal{M}' \models_{\{s\}} P_j \vec{t}_1$. By locality we have $\mathcal{M}' \models_{\{s[\vec{a}/\vec{u}]\}} P_j \vec{t}_1$, i.e. $\mathcal{M}' \models_{\{s[\vec{a}/\vec{u}]\}} (\vec{t}_1 \subseteq \vec{t}_2)'_i$.

Let then $i = j$ and $\vec{a} \in A'_j$. Because $\vec{a} \in X(\vec{t}_1)$, there is $s \in X$ s.t. $s(\vec{t}_1) = \vec{a}$. Since $\mathcal{M} \models_X \vec{t}_1 \subseteq \vec{t}_2$, there is $s' \in X$ s.t. $s'(\vec{t}_2) = s(\vec{t}_1)$. Now $s'(\vec{t}_2) = \vec{a}$, and thus $s'[\vec{a}/\vec{u}](\vec{u}) = s'[\vec{a}/\vec{u}](\vec{t}_2)$, i.e. $\mathcal{M}' \models_{\{s'[\vec{a}/\vec{u}]\}} \vec{u} = \vec{t}_2$. Since $\mathcal{M}' \models_X P_j \vec{t}_1$, by locality and flatness $\mathcal{M}' \models_{\{s'[\vec{a}/\vec{u}]\}} P_j \vec{t}_1$. Thus $\mathcal{M}' \models_{\{s'[\vec{a}/\vec{u}]\}} \vec{u} = \vec{t}_2 \wedge P_j \vec{t}_1$, i.e. $\mathcal{M}' \models_{\{s'[\vec{a}/\vec{u}]\}} (\vec{t}_1 \subseteq \vec{t}_2)'_i$.

Suppose then that there exist $A_1, \dots, A_n \subseteq M^k$ s.t. $\mathcal{M}' \models_X (\vec{t}_1 \subseteq \vec{t}_2)'$, and for each $i \leq n$ and $\vec{a} \in A_i$ there exists $s \in X$ s.t. $\mathcal{M}' \models_{\{s[\vec{a}/\vec{u}]\}} (\vec{t}_1 \subseteq \vec{t}_2)'_i$.

For the sake of proving that $\mathcal{M} \models_X \vec{t}_1 \subseteq \vec{t}_2$, let $s \in X$. Since $\mathcal{M}' \models_X P_j \vec{t}_1$, by flatness we have $\mathcal{M}' \models_{\{s\}} P_j \vec{t}_1$. Now $s(\vec{t}_1) \in P_j^{\mathcal{M}'} = A_j$ and thus there is $s' \in X$ such that $\mathcal{M}' \models_{\{s'[\vec{t}_1]/\vec{u}\}} \vec{u} = \vec{t}_2 \wedge P_j \vec{t}_1$. In particular, $\mathcal{M}' \models_{\{s'[\vec{t}_1]/\vec{u}\}} \vec{u} = \vec{t}_2$, and thus we have $s(\vec{t}_1) = s'[s(\vec{t}_1)/\vec{u}](\vec{u}) = s'[s(\vec{t}_1)/\vec{u}](\vec{t}_2) = s'(\vec{t}_2)$.

- Let $\mu = \psi \wedge \theta$. Suppose first that $\mathcal{M} \models_X \psi \wedge \theta$. Hence $\mathcal{M} \models_X \psi$ and $\mathcal{M} \models_X \theta$. By the inductive hypothesis there are $B_1, \dots, B_n \subseteq M^k$ s.t. $\mathcal{M}[\vec{B}/\vec{P}] \models_X \psi'$ and there are $B'_1, \dots, B'_n \subseteq M^k$ s.t. $\mathcal{M}[\vec{B}'/\vec{P}] \models_X \theta'$. Moreover, for all $i \leq n$ and tuples $\vec{a} \in B_i$ and $\vec{a}' \in B'_i$ there are $s, s' \in X$ s.t. $\mathcal{M}[\vec{B}/\vec{P}] \models_{\{s[\vec{a}/\vec{u}]\}} \psi'_i$ and $\mathcal{M}[\vec{B}'/\vec{P}] \models_{\{s'[\vec{a}'/\vec{u}]\}} \theta'_i$. Let

$$\mathcal{M}' := \mathcal{M}[\vec{A}/\vec{P}], \text{ where } A_i := \begin{cases} B_i & \text{if } P_i \text{ occurs in } \psi' \\ B'_i & \text{if } P_i \text{ does not occur in } \psi' \end{cases}$$

Because none of P_i can occur in both ψ' and θ' , we clearly have $\mathcal{M}' \models_X \psi'$ and $\mathcal{M}' \models_X \theta'$. Hence $\mathcal{M}' \models_X \psi' \wedge \theta'$, i.e. $\mathcal{M}' \models_X (\psi \wedge \theta)'$.

Let $i \leq n$ and let $\vec{a} \in A_i$. Suppose first that P_i occurs in ψ' . Now $\vec{a} \in B_i$, and thus there is $s \in X$ s.t. $\mathcal{M}[\vec{B}/\vec{P}] \models_{\{s[\vec{a}/\vec{u}]\}} \psi'_i$. Relation variables not occurring in ψ' do not occur in ψ'_i either, and thus $\mathcal{M}' \models_{\{s[\vec{a}/\vec{u}]\}} \psi'_i$. Because $(\vec{t}_1 \subseteq \vec{t}_2)_i$ does not occur in θ and $\mathcal{M}' \models_X \theta'$, by Claim I we have $\mathcal{M}' \models_{\{s[\vec{a}/\vec{u}]\}} \theta'_i$. Thus $\mathcal{M}' \models_{\{s[\vec{a}/\vec{u}]\}} \psi'_i \wedge \theta'_i$, i.e. $\mathcal{M}' \models_{\{s[\vec{a}/\vec{u}]\}} (\psi \wedge \theta)'_i$. The case when P_i occurs in θ' is analogous. Finally suppose that P_i does not occur in ψ' nor θ' , whence $(\vec{t}_1 \subseteq \vec{t}_2)_i$ does not occur in $\psi \wedge \theta$. Since $\mathcal{M}' \models_X (\psi \wedge \theta)'$, we can choose any $s \in X$ ($\neq \emptyset$), and then by Claim I we have $\mathcal{M}' \models_{\{s[\vec{a}/\vec{u}]\}} (\psi \wedge \theta)'_i$.

Suppose then that there are $A_1, \dots, A_n \subseteq M^k$ s.t. $\mathcal{M}' \models_X (\psi \wedge \theta)'$, and for every $i \leq n$ and $\vec{a} \in A_i$ there exists $s \in X$ s.t. $\mathcal{M}' \models_{\{s[\vec{a}/\vec{u}]\}} (\psi \wedge \theta)'_i$. Now we have $\mathcal{M}' \models_X \psi' \wedge \theta'$, i.e. $\mathcal{M}' \models_X \psi'$ and $\mathcal{M}' \models_X \theta'$. Because $(\psi \wedge \theta)'_i = \psi'_i \wedge \theta'_i$, for every $i \leq n$ and $\vec{a} \in A_i$ there exists $s \in X$ such that $\mathcal{M}' \models_{\{s[\vec{a}/\vec{u}]\}} \psi'_i$. Since also $\mathcal{M}' \models_X \psi'$, by the inductive hypothesis $\mathcal{M} \models_X \psi$. Analogously we have $\mathcal{M} \models_X \theta$, and thus $\mathcal{M} \models_X \psi \wedge \theta$.

- Let $\mu = \psi \vee \theta$. Suppose first that $\mathcal{M} \models_X \psi \vee \theta$. Thus there are $Y, Y' \subseteq X$ s.t. $Y \cup Y' = X$, $\mathcal{M} \models_Y \psi$ and $\mathcal{M} \models_{Y'} \theta$. By the inductive hypothesis there are $B_1, \dots, B_n, B'_1, \dots, B'_n \subseteq M^k$ s.t. $\mathcal{M}[\vec{B}/\vec{P}] \models_Y \psi'$ and $\mathcal{M}[\vec{B}'/\vec{P}] \models_{Y'} \theta'$. In addition, for every $i \leq n$, $\vec{a} \in B_i$ and $\vec{a}' \in B'_i$ there exist $s \in Y$ and $s' \in Y'$ s.t. $\mathcal{M}[\vec{B}/\vec{P}] \models_{\{s[\vec{a}/\vec{u}]\}} \psi'_i$ and $\mathcal{M}[\vec{B}'/\vec{P}] \models_{\{s'[\vec{a}'/\vec{u}]\}} \theta'_i$. Let

$$\mathcal{M}' := \mathcal{M}[\vec{A}/\vec{P}], \text{ where } A_i := \begin{cases} B_i & \text{if } P_i \text{ occurs in } \psi' \\ B'_i & \text{if } P_i \text{ does not occur in } \psi' \end{cases}$$

Because none of P_i can occur in both ψ' and θ' , we clearly have $\mathcal{M}' \models_Y \psi'$ and $\mathcal{M}' \models_{Y'} \theta'$. Therefore $\mathcal{M}' \models_X \psi' \vee \theta'$, i.e. $\mathcal{M}' \models_X (\psi \vee \theta)'$.

Let $i \leq n$ and $\vec{a} \in A_i$. Suppose first that P_i occurs in ψ' . Now $A_i = B_i$, and thus, by the inductive hypothesis, there is $s \in Y$ ($\subseteq X$) such that $\mathcal{M}[\vec{B}/\vec{P}] \models_{\{s[\vec{a}/\vec{u}]\}} \psi'_i$. Relation variables not occurring in ψ' do not occur in ψ'_i either, and thus $\mathcal{M}' \models_{\{s[\vec{a}/\vec{u}]\}} \psi'_i$, i.e. $\mathcal{M}' \models_{\{s[\vec{a}/\vec{u}]\}} (\psi \vee \theta)'_i$. The case when P_i occurs in θ' is analogous. Suppose then that P_i does not occur in ψ' nor θ' , whence $(\vec{t}_1 \subseteq \vec{t}_2)_i$ does not occur in $\psi \vee \theta$. Since $\mathcal{M}' \models_X (\psi \vee \theta)'$, we can choose any $s \in X$ ($\neq \emptyset$), and then by Claim I we have $\mathcal{M}' \models_{\{s[\vec{a}/\vec{u}]\}} (\psi \vee \theta)'_i$.

Suppose then that there are $A_1, \dots, A_n \subseteq M^k$ s.t. $\mathcal{M}' \models_X (\psi \vee \theta)'$, and for all $i \leq n$ and $\vec{a} \in A_i$ there is $s_{i,\vec{a}} \in X$ s.t. $\mathcal{M}' \models_{\{s_{i,\vec{a}}[\vec{a}/\vec{u}]\}} (\psi \vee \theta)'_i$. Since $\mathcal{M}' \models_X \psi' \vee \theta'$, there are $Y, Y' \subseteq X$ s.t. $Y \cup Y' = X$, $\mathcal{M}' \models_Y \psi'$ and $\mathcal{M}' \models_{Y'} \theta'$. We define the teams Y_i and Y'_i , for every $i \leq n$, and the teams $Z, Z' \subseteq X$:

$$\begin{aligned} Y_i &:= \{s_{i,\vec{a}}[\vec{a}/\vec{u}] \mid \vec{a} \in A_i\} \text{ if } (\vec{t}_1 \subseteq \vec{t}_2)_i \text{ occurs in } \psi, \quad \text{and else } Y_i := \emptyset. \\ Y'_i &:= \{s_{i,\vec{a}}[\vec{a}/\vec{u}] \mid \vec{a} \in A_i\} \text{ if } (\vec{t}_1 \subseteq \vec{t}_2)_i \text{ occurs in } \theta, \quad \text{and else } Y'_i := \emptyset. \\ Z &:= Y \cup \left(\bigcup_{i \leq n} Y_i \upharpoonright \text{dom}(X) \right), \quad Z' := Y' \cup \left(\bigcup_{i \leq n} Y'_i \upharpoonright \text{dom}(X) \right). \end{aligned}$$

We then show that for every $i \leq n$ it holds that $\mathcal{M}' \models_{Y_i} \psi'_i$. Let $i \leq n$. If $(\vec{t}_1 \subseteq \vec{t}_2)_i$ does not occur in ψ , then $Y_i = \emptyset$ whence trivially $\mathcal{M}' \models_{Y_i} \psi'_i$. Suppose then that $(\vec{t}_1 \subseteq \vec{t}_2)_i$ occurs in ψ . Now $(\psi \vee \theta)'_i = \psi'_i$ and thus $\mathcal{M}' \models_{\{r\}} \psi'_i$ for every $r \in Y_i$. By flatness we thus have $\mathcal{M}' \models_{Y_i} \psi'_i$. Since $\mathcal{M}' \models_Y \psi'$ and $\mathcal{M}' \models_{Y_i} \psi'_i$ for every $i \leq n$, we can apply Claim II for each $i \leq n$ to obtain $\mathcal{M}' \models_Z \psi'$. By a symmetric argumentation $\mathcal{M}' \models_{Z'} \theta'$.

We then show that $\mathcal{M} \models_Z \psi$. We may suppose that $Z \neq \emptyset$, since else trivially $\mathcal{M} \models_Z \psi$. Let $i \leq n$ and let $\vec{a} \in A_i$. Suppose first that $(\vec{t}_1 \subseteq \vec{t}_2)_i$ occurs in ψ . Now $(\psi \vee \theta)'_i = \psi'_i$ and thus there is $s_{i,\vec{a}} \in X$ s.t. $\mathcal{M}' \models_{\{s_{i,\vec{a}}[\vec{a}/\vec{u}]\}} \psi'_i$. By the definition of Y_i we must have $s_{i,\vec{a}}[\vec{a}/\vec{u}] \in Y_i$ and moreover $s_{i,\vec{a}} \in Z$. Suppose then that $(\vec{t}_1 \subseteq \vec{t}_2)_i$ does not occur in ψ . Then we can choose any assignment $s \in Z$ ($\neq \emptyset$), whence by Claim I we have $\mathcal{M}' \models_{\{s[\vec{a}/\vec{u}]\}} \psi'_i$. Hence, by the inductive hypothesis, $\mathcal{M} \models_Z \psi$. We can analogously deduce $\mathcal{M} \models_{Z'} \theta$. Since $Z \cup Z' = X$, we have $\mathcal{M} \models_X \psi \vee \theta$.

- Let $\mu = \exists x \psi$. Suppose first that $\mathcal{M} \models_X \exists x \psi$, i.e. there is $F : X \rightarrow \mathcal{P}^*(M)$, s.t. $\mathcal{M} \models_{X[F/x]} \psi$. By the inductive hypothesis there are $A_1, \dots, A_n \subseteq M^k$ s.t. $\mathcal{M}' \models_{X[F/x]} \psi'$, where $\mathcal{M}' := \mathcal{M}[\vec{A}/\vec{P}]$. Moreover, for all $i \leq n$ and $\vec{a} \in A_i$ there is $r \in X[F/x]$ s.t. $\mathcal{M}' \models_{\{r[\vec{a}/\vec{u}]\}} \psi'_i$. Since $\mathcal{M}' \models_{X[F/x]} \psi'$, we have $\mathcal{M}' \models_X \exists x \psi'$, i.e. $\mathcal{M}' \models_X (\exists x \psi)'$.

Let $i \leq n$ and let $\vec{a} \in A_i$. Now there is $r \in X[F/x]$ s.t. $\mathcal{M}' \models_{\{r[\vec{a}/\vec{u}]\}} \psi'_i$. Since $r \in X[F/x]$, there is $s \in X$ and $b \in F(s)$ s.t. $r = s[b/x]$. Let $F' :$

$\{s[\vec{a}/\vec{u}]\} \rightarrow \mathcal{P}^*(M)$ s.t. $s[\vec{a}/\vec{u}] \mapsto \{b\}$. Since $\{s[\vec{a}/\vec{u}]\}[F'/x] = \{r[\vec{a}/\vec{u}]\}$, we have $\mathcal{M}' \models_{\{s[\vec{a}/\vec{u}]\}} \exists x \psi'_i$, i.e. $\mathcal{M}' \models_{\{s[\vec{a}/\vec{u}]\}} (\exists x \psi)_i'$.

Suppose then that there are $A_1, \dots, A_n \subseteq M^k$ s.t. $\mathcal{M}' \models_X (\exists x \psi)'$, and for every $i \leq n$ and $\vec{a} \in A_i$ there is $s_{i,\vec{a}} \in X$ s.t. $\mathcal{M}' \models_{\{s_{i,\vec{a}}[\vec{a}/\vec{u}]\}} (\exists x \psi)_i'$. Now there is $F : X \rightarrow \mathcal{P}^*(M)$ s.t. $\mathcal{M} \models_{X[F/x]} \psi'$. Furthermore, for each $i \leq n$ and $\vec{a} \in A_i$ there is $F_{i,\vec{a}} : \{s_{i,\vec{a}}[\vec{a}/\vec{u}]\} \rightarrow \mathcal{P}^*(M)$ s.t. $\mathcal{M}' \models_{\{s_{i,\vec{a}}[\vec{a}/\vec{u}]\}[F_{i,\vec{a}}/x]} \psi'_i$. For each $i \leq n$ let

$$X'_i := \bigcup_{\vec{a} \in A_i} \{s_{i,\vec{a}}[\vec{a}/\vec{u}]\}[F_{i,\vec{a}}/x].$$

By flatness $\mathcal{M}' \models_{X'_i} \psi'_i$ for each $i \leq n$. Let $F' : X \rightarrow \mathcal{P}^*(M)$ s.t.

$$s \mapsto F(s) \cup \{b \in F_{i,\vec{a}}(s_{i,\vec{a}}[\vec{a}/\vec{u}]) \mid i \leq n, \vec{a} \in A_i \text{ s.t. } s = s_{i,\vec{a}}\}.$$

By the definitions of F' and X'_i ($i \leq n$) we have

$$X[F/x] \cup \left(\bigcup_{i \leq n} X'_i \upharpoonright \text{dom}(X[F/x]) \right) = X[F'/x].$$

Thus by applying Claim II for each $i \leq n$, we obtain $\mathcal{M}' \models_{X[F'/x]} \psi'$. Moreover, now for each $i \leq n$ and $\vec{a} \in A_i$ there is $r \in X[F'/x]$ s.t. $\mathcal{M}' \models_{\{r[\vec{a}/\vec{u}]\}} \psi'_i$. Thus, by the inductive hypothesis, $\mathcal{M} \models_{X[F'/x]} \psi$, i.e. $\mathcal{M} \models_X \exists x \psi$.

- Let $\mu = \forall x \psi$. Suppose first that $\mathcal{M} \models_X \forall x \psi$, i.e. $\mathcal{M} \models_{X[M/x]} \psi$. By the inductive hypothesis there are $A_1, \dots, A_n \subseteq M^k$ s.t. $\mathcal{M}' \models_{X[M/x]} \psi'$. Moreover, for each $i \leq n$ and $\vec{a} \in A_i$ there exists $r \in X[M/x]$ s.t. $\mathcal{M}' \models_{\{r[\vec{a}/\vec{u}]\}} \psi'_i$. Now $\mathcal{M}' \models_X \forall x \psi'$, i.e. $\mathcal{M}' \models_X (\forall x \psi)'$.

Let $i \leq n$ and let $\vec{a} \in A_i$. Now there is $r \in X[M/x]$ s.t. $\mathcal{M}' \models_{\{r[\vec{a}/\vec{u}]\}} \psi'_i$. Since $r \in X[M/x]$, there is $s \in X$ and $b \in M$ s.t. $r = s[b/x]$. Let $F : \{s[\vec{a}/\vec{u}]\} \rightarrow \mathcal{P}^*(M)$ s.t. $s[\vec{a}/\vec{u}] \mapsto \{b\}$. Now $\{s[\vec{a}/\vec{u}]\}[F/x] = \{r[\vec{a}/\vec{u}]\}$, and therefore $\mathcal{M}' \models_{\{s[\vec{a}/\vec{u}]\}} \exists x \psi'_i$. Since $\mathcal{M}' \models_X \forall x \psi'$, by flatness and locality we have $\mathcal{M}' \models_{\{s[\vec{a}/\vec{u}]\}} \forall x \psi'$. Hence $\mathcal{M}' \models_{\{s[\vec{a}/\vec{u}]\}} \exists x \psi'_i \wedge \forall x \psi'$, i.e. $\mathcal{M}' \models_{\{s[\vec{a}/\vec{u}]\}} (\forall x \psi)_i'$.

Suppose then that there are $A_1, \dots, A_n \subseteq M^k$ s.t. $\mathcal{M}' \models_X (\forall x \psi)'$, and that for each $i \leq n$ and $\vec{a} \in A_i$ there is $s \in X$ s.t. $\mathcal{M}' \models_{\{s[\vec{a}/\vec{u}]\}} (\forall x \psi)_i'$. Now we have $\mathcal{M}' \models_X \forall x \psi'$, i.e. $\mathcal{M}' \models_{X[M/x]} \psi'$.

Let $i \leq n$ and let $\vec{a} \in A_i$. Now there is $s \in X$ s.t. $\mathcal{M}' \models_{\{s[\vec{a}/\vec{u}]\}} \exists x \psi'_i \wedge \forall x \psi'$ and thus there is $F : \{s[\vec{a}/\vec{u}]\} \rightarrow \mathcal{P}^*(M)$ s.t. $\mathcal{M}' \models_{\{s[\vec{a}/\vec{u}]\}[F/x]} \psi'_i$. Let $b \in F(s[\vec{a}/\vec{u}])$ and let $r := s[b/x]$, whence $r \in X[M/x]$ and $r[\vec{a}/\vec{u}] \in \{s[\vec{a}/\vec{u}]\}[F/x]$. Now by flatness $\mathcal{M}' \models_{\{r[\vec{a}/\vec{u}]\}} \psi'_i$. Therefore, by the inductive hypothesis, $\mathcal{M} \models_{X[M/x]} \psi$, i.e. $\mathcal{M} \models_X \forall x \psi$. \square